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Conditions for the Existence Of Taylor's Theorem

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CONDITIONS FOR THE EXISTENCE OF TAYLOR'S THEOREM

by

Maud M. Patterson.

Thesis for the Degree of A.B.
in the
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of the
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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

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Introduction

General statement of the Problem: - The problem with which this thesis is concerned is the conditions for the existence of Taylor's theorem; that is for the existence of the series

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + \cdots\cdots + f^{n-1}(x)\frac{h^{n-1}}{(n-1)!} + f^{n}(x+\theta h)\frac{h^n}{n!}.$$

In the first chapter, we have discussed the proofs of the theorem, which are given in many of the standard treatises on calculus. In the second chapter, we have considered a few examples where the theorem does not hold, and also a necessary and sufficient condition for the existence of

Taylor's theorem.

Historical Sketch of Taylor and the Development of Taylor's Theorem:

Brook Taylor, an English mathematician and a great admirer of Newton, was born at Edmonton in 1685, and died at London in 1731. He was educated at St. John's College, Cambridge. In 1712, he was elected a fellow of the Royal Society, and acted as secretary of the society from 1714 until he resigned this position in 1719. From 1712, he wrote papers for the Philosophical Transactions in which he discussed motion of projectiles, centre of oscillation, and forms of liquids raised by capillarity. In 1715, Taylor published his Methodus Incrementorum Directa et Inversa, which contained the proof of

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots,$$

by which any function of a single

variable can be expanded in terms of this variable. The expansion is called Taylor's theorem. Development of functions by infinite series were first promulgated during the latter part of the seventeenth century. The first series published were by the expansion of $\log(1+x)$, known as the area of the hyperbola, by Nicholas Mercator in 1668. In a letter of James Gregory to J. Collins in 1671, he discussed the expansion of arcs and tangents of the tangent. Newton directed his attention in this direction, and he arrived at the binomial theorem, and other general expansions, such as $\sin x$, $\cos x$, x^x, etc. In 1715, Taylor published his expansion, which may be written thus

$$\int y\,dx = xy - \frac{x}{2!}\frac{dy}{dx} + \frac{x^3}{3!}\frac{d^2y}{dx^2} - \cdots$$

Taylor commenced to work on the

expansion of an infinite series, and he found that all of these expansions were but particular cases of the general theorem known as Taylor's theorem.

Taylor's Original Proof:-

Introducing the modern notation, Taylor's proof of the theorem, with some modifications is as follows:

Let $f(x)$ be any function of x, and suppose x changes successively into $x+\Delta x, x+2\Delta x, x+3\Delta x, \cdots\cdots, x+n\Delta x$, and let the functions

$f(x), f(x+\Delta x), f(x+2\Delta x), \cdots\cdots; f(x+n\Delta x)$

be represented by $y, y_1, y_2, y_3, \cdots\cdots y_n$.

Then we have

$y_1 - y = \Delta y,\ y_2 - y_1 = \Delta y_1, \cdots\cdots y_n - y_{n-1} = \Delta y_{n-1}$

$\Delta y_1 - \Delta y = \Delta^2 y,\ \Delta y_2 - \Delta y_1 = \Delta^2 y_1, \cdots \Delta y_n - \Delta y_{n-1} = \Delta^2 y_{n-1}$.

$\Delta^2 y_1 - \Delta^2 y = \Delta^3 y$ etc.

The final result we wish to express, y_n in terms of $y, \Delta y, \Delta^2 y \cdots\cdots \Delta^n y$.

Then we have

$y_n = y_{n-1} + \Delta y_{n-1} = y_{n-2} + 2\Delta y_{n-2} + \Delta^2 y_{n-2}$.

In a similar manner by substitution, $y_{n-3} + \Delta y_{n-3}$ for y_{n-2}, we have

$y_n = y_{n-3} + 3\Delta y_{n-3} + 3\Delta^2 y_{n-3} + \Delta^3 y_{n-3}$.

The coefficients are the same in this equation as they are in the expansion of $(a+b)^3$. Now if it assumed that the same law holds for any value of n, we readily see that

$$y_n = y + n\Delta y + \frac{n(n-1)}{2!}\Delta^2 y + \frac{n(n-1)(n-2)}{3!}\Delta^3 y + \cdots + \Delta^n y.$$

Regarding Δ as a symbol of operation, the equation $y_n = y_{n-1} + \Delta y_{n-1}$ may be written $y_n = (1+\Delta)\, y_{n-1}$.

In a similar manner

$$y_{n-1} = (1+\Delta)\, y_{n-2}$$

$$\therefore y_n = (1+\Delta)(1+\Delta)\, y_{n-2} = (1+\Delta)^2 y_{n-2}.$$

Also

$$y_n = (1+\Delta)^3 y_{n-3}$$

and in general

$$y_n = (1+\Delta)^n y$$

$$= \left(1 + n\Delta + \frac{n(n-1)}{2!}\Delta^2 + \cdots\cdots + \Delta^n\right) y.$$

$$y_n = y + n\Delta y + \frac{n(n-1)}{2!}\Delta^2 y + \cdots + \Delta^n y$$

If we place $n, \Delta x = h$ or $n = \frac{h}{\Delta x}$, the above equation becomes

$$f(x+h) = y + h\frac{\Delta y}{\Delta x} + \frac{h(h-\Delta x)}{2!}\frac{\Delta^2 y}{\Delta x^2} + \cdots$$

If now, h being regarded as constant, we suppose Δx to decrease, and consequently Δx to decrease indefinitely, we obtain on proceeding to the limit the following equation.

$$f(x+h) = y + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2 y}{dx^2} + \cdots$$

$$= f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$$

This is Taylor's theorem. He does not write the conditions under which this series can be applied, i.e. the remainder after n terms. He does mention that "the function $f(x)$ and all of its derived functions $f'(x)$, $f''(x)$, $f'''(x)$ --- must be finite and continuous between the limits

the infinite series may equal the value of $f(x+h)$.

Taylor did not seem to be aware of the importance of his theorem, and did not call attention to its application. This theorem is the foundation stone of the entire superstructure of the differential calculus. Taylor's series was the result of his investigations, and he was the first to show how to expand a function in a series of ascending powers of the variable. He was the founder of the differential calculus, and was the first person to find an expression for the remainder of the series after any number of terms.

Lagrange's Proof of the Remainder: Let R_n be the remainder after n terms in Taylor's theorem, and then let $X = x + j$, and we have

$$f(X) = f(x) + (X-x)f'(x) + \frac{(X-x)^2}{2!}f''(x) + \cdots\cdots + \frac{(X-x)^{n-1}}{(n-1)!}f^{n-1}(x) + R_n, \qquad (1)$$

in which $f(x)$, $f'(x)$, $f''(x)$ ... $f^{n-1}(x)$ are finite and continuous for all values of the variable between X and x.

Let $$R_n = \frac{(X-x)^n}{n!} P,$$

where P is some function of X and x.

Then,

$$f(X) - \left[f(x) + (X-x) f'(x) + \cdots + \frac{(X-x)^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{(X-x)^n}{n!} P \right] = 0 \qquad (2)$$

Substitute z for x in (2), and let $F(z)$ represent the resulting function

$$F(z) = f(X) - \left[f(z) + (X-z) f'(z) + \cdots + \frac{(X-z)^n}{n!} P \right] (2)$$

The right hand side in this equation vanishes when $z = X$

$$\therefore \quad F(X) = 0.$$

Also from (2) the right hand side vanishes when $z = x$

$$\therefore \quad F(x) = 0.$$

Since $F(z)$ vanishes when $z = X$ and when $z = x$, then $F'(z)$ vanishes for some value of z between X and x. This follows

by means of Rolle's Theorem*

By differentiating (3) we have

$$F'(z) = \frac{-(X-z)^{n-1}}{(n-1)!} f^n(z) + \frac{(X-z)^{n-1}}{(n-1)!} P$$

Consequently for some value of z between x and X, we must have $f^n(z) = P$.

If θ is a proper fraction and $\lambda + \theta(X-\lambda)$, by assuming a suitable value for θ, can be made equal to any quantity between λ and X. Then

$$P = f^n[\lambda + \theta(X-\lambda)]$$

and $$R_{n-1} = \frac{(X-\lambda)^n}{n!} f^n[\lambda + \theta(X-\lambda)]$$

Substituting this value of R_{n-1} in (1), we have

$$f(X) = f(\lambda) + (X-\lambda) f'(\lambda) + \frac{(X-\lambda)^2}{2!} f''(\lambda) + \cdots + \frac{(X-\lambda)^{n-1}}{(n-1)!} f^{n-1}(\lambda) + \frac{(X-\lambda)^n}{n!} f^n[\lambda + \theta(X-\lambda)]$$

If h is substituted for $(X-\lambda)$, we have

---

* Chapter I, § 1, page 11 of this thesis.

$$f(x+h) = f(x) + h f'(x) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x)$$

$$+ \frac{h^n}{n!} f^n(x + \theta h).$$

The essential conditions for the application of Taylor's theorem for the expansion of any function in a series are that none of its derived functions shall become infinite, and that $\frac{h^n}{n!} f^n(x+\theta h)$ shall become infinitely small as n becomes sufficiently large.

Chapter I

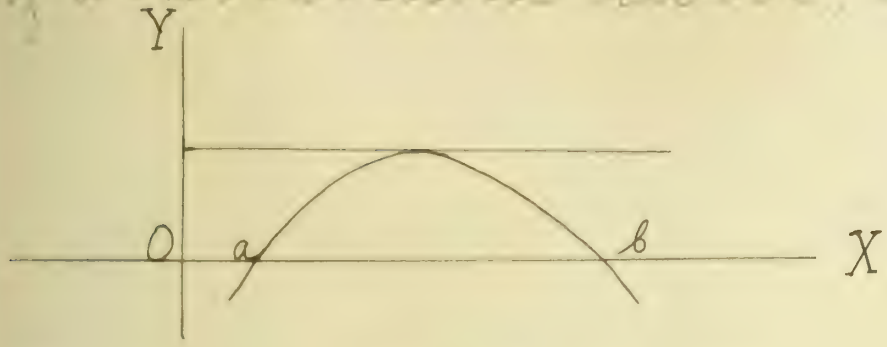
Proof of Taylor's Theorem.

§1. Rolle's Theorem. — If $f(x)$ and its first $n+1$ derivatives are continuous for all values of x between a and b, and if $f(a)$ and $f(b)$ both vanish, then $f'(x)$ will vanish for some value of x between a and b.

By supposition $f(x)$ cannot become infinite for any value of x, where $a<x<b$; and if $f'(x)$ does not vanish, it must always be positive or always be negative. Hence $f(x)$ must continually increase or continually decrease. This is impossible, as $f(a)=0$ and $f(b)=0$, hence at some point x between a and b, $f(x)$ must cease to increase and begin to decrease. This point is

defined by the equation $f'(x) = 0$.

To prove the same thing geometrically, let $y = f(x)$ be the equation of a continuous curve, which crosses



the x axis at the distances $x = a$ and $x = b$ from the origin; then at some point between a and b the tangent to the curve is parallel to the x axis, since by hypothesis there is no discontinuity in the direction of the tangent. Hence

$$\frac{dy}{dx} = f'(x) = 0.$$

This theorem is used in many cases in getting the general form of the remainder for Taylor's theorem.

§2 Classification of the Proofs of Taylor's Theorem. — The treatises on calculus develop Taylor's theorem in many different ways, among which are the following:

1. Development based upon undetermined coefficients. In this case we assume that a function can be expanded in powers of a variable, in which the coefficients are functions of another variable. Upon determining the values of these coefficients, we have Taylor's series. Such proofs are given by the following:

Mc Mahon and Snyder.
Williamson
Newcomb.
Nichols
Rice and Johnson.
Bass.
Edwards.
Hayes

Perry
Courtenay
Buckingham
Byerly
* all
Church
Bowser
* ardy
Lamb 1st proof
Osborne 1st proof
Young and Linebarger
Olney
Todhunter 1st proof
Fricke
Kiepert

2. Development based upon the* law of the mean or Rolle's theorem. Some of these proofs are developed by the simple law of the mean, and others by the use of the generalized

* By the law of the mean we understand the generalization of Rolle's theorem.

law of the means. All of these proofs use the Rolle's theorem at some point of the proof, but there is one sort of proof which begins at once with the application of this theorem. These methods are used by the following:

Lambert
Harnack
Price
Osborne 2nd proof
Miller
Todhunter 2nd proof
Serret-Bohlmann.

3. Development based upon the binomial theorem. The expansion is developed first for a finite number of terms by the processes of Algebra and then extended to an infinite series. This method of development is used by the following:

Genocchi
Goursat

Laurent

7. Development based upon the laws of integration.

Lamb 2nd proof.

Picard

Jordan

Walton 1st proof

5. Miscellaneous proofs.

Walton. 2nd proof.

Kennedy

Lamb 3rd proof.

Stolz

§ 3. Discussion of Proofs of Taylor's Theorem. The following proof is one which involves undetermined coefficients.

*Let us assume that the function $f(x+y)$ can be expanded in powers of y.

$$f(x+y) = P_0 + P_1 y + P_2 y^2 + \cdots\cdots + P_n y^n + \cdots\cdots$$

in which $P_0, P_1, P_2, \cdots\cdots P_n \cdots$ are finite and continuous functions of x. When $y=0$, the expansion reduces to $f(x) = P_0$.

Let $u = f(x+y)$.

* Williamson: Differential Calculus. p 61.

By differentiating with respect to x we have

$$\frac{\partial u}{\partial x} = \frac{dP_0}{dx} + y\frac{dP_1}{dx} + y^2\frac{dP_2}{dx} + \cdots + y^n\frac{dP_n}{dx} + \quad (2)$$

By differentiating with respect to y, we have

$$\frac{\partial u}{\partial y} = P_1 + 2P_2 y + 3P_3 y^2 + \cdots + nP_n y^{n-1} + \cdots$$

In order that these series be identical for all values of y, the coefficients of like powers must be equal. Hence, we must have

$$P_1 = \frac{dP_0}{dx} = \frac{df(x)}{dx} = f'(x)$$

$$P_2 = \frac{1}{2}\frac{dP_1}{dx} = \frac{1}{2!}\frac{d^2f(x)}{dx^2} = \frac{1}{2!}f''(x)$$

$$P_3 = \frac{1}{3}\frac{dP_2}{dx} = \frac{1}{3!}\frac{d^3f(x)}{dx^3} = \frac{1}{3!}f'''(x)$$

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$$P_n = \frac{1}{n}\frac{dP_{n-1}}{dx} = \frac{1}{n!}\frac{d^nf(x)}{dx^n} = \frac{1}{n!}f^n(x)$$

- - - - - - - - - - - - - - - - - -

If $f(x)$ and its successive derived functions are finite and continuous, then by replacing $P_1, P_2 \cdots$

R_n by these values, we have

$$f(x+y) = f(x) + y f'(x) + \frac{y^2}{2!} f''(x) + \frac{y^3}{3!} f'''(x) +$$

$$\cdots\cdots + \frac{y^n}{n!} f^n(x) + \cdots\cdots \qquad (1)$$

It is now necessary to find the expression for sum of the series after the first n terms, in order to tell whether the series converges. Suppose R_n represents the remainder after n terms in Taylor's expansion. Then by writing X for $x+y$ in (1), we have

$$f(X) = f(x) + (X-x) f'(x) + \frac{(X-x)^2}{2!} f''(x) + \cdots\cdots$$

$$+ \frac{(X-x)^n}{n!} f^n(x) + R_n \qquad (2)$$

in which $f(x)$ and its derivatives are finite and continuous for all values of the variable between X and x. From the form of terms included in R_n, we may write

$$R_n = \frac{(X-x)^n}{n!} P$$

where P is some function of X and x

to be determined. We have then

$$f(X) - \left[f(x) + (X-x) f'(x) + \frac{(X-x)^2}{2!} f''(x) + \cdots \right.$$

$$\left. + \frac{(X-x)^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{(X-x)^n}{n!} P \right] = 0 \qquad (3)$$

Let z be substituted for x in every term except P, and let $F(z)$ represent the resulting expression; we have

$$F(z) = f(X) - \left[f(z) + (X-z) f'(z) + \cdots + \frac{(X-z)^n}{n!} P \right] \qquad (4)$$

in which P has the same value as before. The right hand side of (4) vanishes when $z = X$ $\therefore F(X) = 0$

From (2), the right hand side of (4) vanishes when $z = x$ $\therefore F(x) = 0$.

Since $F(z)$ vanishes when $z = X$, and when $z = x$, then by Rolle's theorem, its derived function $F'(z)$ must vanish for some value of the variable between X and x. By differentiating (4) we have

$$F'(z) = -\frac{(X-z)^{n-1}}{(n-1)!} f^n(z) + \frac{(X-z)^{n-1}}{(n-1)!} P$$

Consequently for some value of z between X and x, we must have $f^n(z) = 0$.

If θ is a positive quantity less than unity, it is seen that $x + \theta(X - x)$, by assigning a suitable value to θ can be made equal to any number between X and x.

$$\therefore \quad P = f^n[x + \theta(X - x)] \quad ; \quad 0 < \theta < 1$$

The remainder, after n terms of Taylor's series, is then

$$R_n = \frac{(X - x)^n}{n!} f^n[x + \theta(X - x)]$$

Substituting the value of R_n in (2), we have:

$$f(X) = f(x) + (X - x) f'(x) + \frac{(X - x)^2}{2!} f''(x) + \cdots$$

$$+ \frac{(X - x)^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{(X - x)^n}{n!} f^n[x + \theta(X - x)].$$

Substituting h for $(X - x)$, we have

$$f(x + h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots$$

$$+ \frac{h^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{h^n}{n!} f^n(x + \theta h).$$

Conditions for the expansion of any function in a series consisting of an infinite number of terms are that none of its derived functions shall become infinite, and that the quantity $\frac{h^n}{n!} f^n(x+\theta h)$ shall become infinitely small, when n is taken sufficiently large; otherwise the series does not admit of a finite limit, and it does not converge.

In this proof the equation (α) need not necessarily be true, except when we have the terms $P_0, P_1, P_2 \ldots P_{n-1}$ and also their derivates, finite and continuous. In the proof nothing was said as to whether these terms $P_0, P_1, P_2, \ldots P_n \ldots$ are finite and continuous.

Many of the proofs similar to the one above do not obtain a remainder after a finite number of terms, some do not consider the

convergency of the series, and others do not consider whether $f(x)$ and its successive derivatives are finite and continuous. For any of these cases the proofs are faulty.

Mc Mahon and Snyder gives a proof similar to the one just given, in which Maclaurin's expansion is first developed, and then Taylor's series is developed.

*We are to find a formula for the expansion in ascending powers of x of any function, which with its derivatives are continuous in the vicinity of $x=0$. We will assume that the function $f(x)$ has such a development, and that the latter can be treated as identical with the former for all values of x within a certain interval of equivalence that includes $x=0$

*Mc Mahon and Snyder: Differential Calculus p 87

Let us assume the identity

$$f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots \qquad (1)$$

Then

$$f'(x) = B + 2Cx + 3Dx^2 + 4Ex^3 + \cdots$$

$$f''(x) = 2C + 2.3Dx + 4.3Ex^2 + \cdots$$

$$f'''(x) = 2.3D + 4.3.2Ex + \cdots$$

$$f^{IV}(x) = 4.3.2E + \cdots$$

in which by hypothesis x may have any value within a certain interval including $x = 0$. Substitute 0 for x in the identities and we have

$$f(0) = A,\ f'(0) = B,\ f''(0) = 2C,\ f'''(0) = 2.3D \cdots$$

hence $A = f(0),\ B = f'(0),\ C = \frac{f''(0)}{2!},\ D = \frac{f'''(0)}{3!} \cdots$

Substituting these values in (1), we have

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$+ \frac{f^n(0)}{n!}x^n + \cdots \qquad (2)$$

This series is known as Maclaurin's series. The conditions that a series may give the true value of a function are that $f(x)$, $f'(x)$, $f''(x)$ ....

are finite and continuous in the vicinity of $x=0$, and that the values of x be restricted to lie within a certain interval, not wider than the interval of convergence of the series. If $f(x)$ or any of its derivatives are discontinuous in the vicinity of $x=0$, then $f(x)$ has no development in the powers of x. If, however, these successive derivatives are continuous in the vicinity of some value $x=a$, then $f(x)$ will have a development in powers of $x-a$, which will represent the function for values of x within a certain interval in the vicinity of $x=a$.

We will assume the expression

$$f(x) = A + B(x-a) + C(x-a)^2 + D(x-a)^3 + E(x-a)^4 + \cdots \quad (3).$$

With the same hypothesis in the vicinity of $x=a$, as before in the vicinity of $x=0$, we have by differentiation of (3)

$$f'(x) = B + 2C(x-a) + 3D(x-a)^2 + 4E(x-a)^3 + \cdots$$

$$f''(x) = 2C + 3.2D(x-a) + 4.3E(x-a)^2 + \cdots$$

$f'''(x) = 3.2D + 4.3.2E(x-a) + \cdots\cdots$

If the value a be given to x we obtain
$f(a) = A$, $f'(a) = B$, $f''(a) = 2C$, $f'''(a) = 2.3.4 \cdots\cdots$
Hence $A = f(a)$, $B = f'(a)$, $C = \frac{f''(a)}{2!}$, $D = \frac{f'''(a)}{3!} \cdots\cdots$

Substituting these values in (3) we have:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$
$$+ \cdots\cdots + \frac{f^n(a)}{n!}(x-a)^n + \cdots\cdots \quad (4)$$

Let the remainder after n terms be denoted by $P_n(x, a)$, which is a function of x and a as well as of n. We will write

$$P_n(x, a) = \frac{(x-a)^n}{n!}\phi(x, a),$$

and then we must determine $\phi(x, a)$, so that the following will be an algebraic identity.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots\cdots$$
$$+ \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1} + \frac{\phi(x, a)}{n!}(x-a)^n \quad (5)$$

We will have

$$f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 - \cdots$$

$$- \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1} - \frac{\phi(x,a)}{n!}(x-a)^n = 0 \qquad (6)$$

Let us define a new function $F(z)$ as follows:

$$F(z) = f(x) - f(z) - f'(z)(x-z) - \frac{f''(z)}{2!}(x-z)^2 - \cdots$$

$$- \frac{f^{n-1}(z)}{(n-1)!}(x-z)^{n-1} - \frac{\phi(x,a)}{n!}(x-z)^n, \qquad (7)$$

in which the right hand member is obtained from (6) by replacing a by the variable z in every term except $\phi(x,a)$. $F(z)$ vanishes when $z = x$, and by (6), when $z = a$. Hence by Rolle's theorem, $F'(z)$ vanishes for some value of z, say z_1, between x and a. By differentiating (7) we get the following equation:

$$F'(z) = -f'(z) + f'(z) - f''(z)(x-z) + f''(z)(x-z) - \cdots$$

$$- \frac{f^n(z)}{(n-1)!}(x-z)^{n-1} + \frac{\phi(x,a)}{(n-1)!}(x-z)^{n-1},$$

and all these terms but the last two vanish. Therefore

$$F'(z) = \frac{(x-z)^{n-1}}{(n-1)!}\left[\phi(x,a) - f^n(z)\right]$$

Since $F'(z)$ vanishes when $z = z_1$, then

$$\phi(x,a) = f^n(z_1) \qquad (8)$$

where z_1 lies between x and a, and may be represented by $z_1 = a + \theta_1(x-a)$, where θ is a proper fraction.

Hence from (8)

$$\phi(x,a) = f^n[a + \theta(x-a)]$$

and $R_n(x,a) = \dfrac{f^n[a+\theta(x-a)]}{n!}(x-a)^n$

The complete form of the expansion will then be the following:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots\cdots$$

$$+ \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1} + \frac{f^n[a+\theta(x-a)]}{n!}(x-a)^n \qquad (9)$$

in which n is any positive integer.

We now have an expression for $f(x)$ in powers of $(x-a)$, and if x be every where replaced by $(x+a)$, then $(x-a)$ will be replaced by x, and the identity will be

$$f(x+a) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \cdots + \frac{f^{n-1}(a)}{(n-1)!}x^{n-1}$$

$$+ \frac{f^n(a+\theta x)}{n!}x^n \qquad (10)$$

in which x and a are any two numbers, n a positive integer and θ some proper fraction which may or not, however, be independent of the values of the other letters. The equation (11) is the form of Taylor's series.

A proof based on the law of the mean: *Law of the Mean. If f(x) is a function in the interval from a to b, whose progressive and regressive differential quotients are every where in the interval identical and determinate, then a value x_1 can be found between a and b, such that we shall have a relation of differences

$$\frac{f(b) - f(a)}{b - a} = f'(x_1)$$

Denote the value of the quotient of differences by k. and we shall then have:

*Harnack: Introduction to Calculus p 4:

$$[f(b) - kb] - [f(a) - ka] = 0$$

Define the function $\phi(x)$ thus:

$$\phi(x) = [f(b) - kb] - [f(x) - kx]$$

Then $\phi(x)$ and $f(x)$ is continuous and will likewise have ~~the~~ identical ~~fun~~ and ~~expressed~~ derivatives ~~quotient~~. Furthermore it will have the value zero for $x = a$ and $x = b$. Hence there must be, in the interval (a, b), a value x_1, which will make $\phi'(x_1) = 0$. We have

$$\phi'(x_1) = k - f'(x_1) = 0$$

$$\therefore \quad k = \frac{f(b) - f(a)}{b - a} = f'(x_1).$$

Let $a < x_1 < b$, then $x_1 = a + \theta(b - a)$, where θ denotes a positive proper fraction and we will have

$$\frac{f(b) - f(a)}{b - a} = f'[a + \theta(b - a)], \quad 0 < \theta < 1.^*$$

Let $f(x)$ be a function in the interval (a, b) and let its derived functions

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*Harnack: Introduction to Calculus p 65

$f'(x)$, $f''(x)$ ... $f^{n-1}(x)$ be everywhere in the same interval finite and continuous. We must first see if we can [illegible] so $\frac{f(b) - f(a) - (b-a)f'(a)}{(b-a)^2}$, which may be denoted by k, by means of higher derivatives. We have

$$f(b) - f(a) - (b-a)f'(a) - k(b-a)^2 = 0$$

Define $\phi(x)$ in the following way:

$$\phi(x) = f(b) - f(x) - (b-x)f'(x) - (b-x)^2 k,$$

which is a continuous function vanishing for $x = a$ and $x = b$. There must be some value x_1 such that

$$\phi'(x_1) = -f'(x_1) + f'(x_1) - (b-x_1)f''(x_1) + 2(b-x_1)k = 0$$

$$\therefore \quad k = \tfrac{1}{2} f''(x_1).$$

Accordingly, we have

$$f(b) = f(a) + (b-a)f'(a) - \tfrac{1}{2}(b-a)^2 f''[a + \theta(b-a)]$$

for $0 < \theta < 1$.

Proceed in a similar way and put

$$f(b) - f(a) - (b-a)f'(a) - \tfrac{1}{2}(b-a)^2 f''(a) - \tfrac{1}{2!}(b-a)^3 k = 0.$$

Then $\quad k = \tfrac{1}{3} f'''(x)$

$$\therefore f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''[a + \theta(b-a)]$$

If we proceed in a similar way for any value n, we have

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(b-a)^p}{p}k.$$

where p is any positive integer.

Define $\phi(x)$ as the following expression:

$$\phi(x) = f(b) - f(x) - f'(x)(b-x) - \frac{(b-x)^2}{2!}f''(x) - \cdots - \frac{(b-x)^{n-1}}{(n-1)!}f^{n-1}(x) - \frac{(b-x)^p}{p}k.$$

$\phi(x)$ is finite and continuous and vanishes at $x=a$ and $x=b$, so we have.

$$\phi'(x_1) = \frac{(b-x_1)^{n-1}}{(n-1)!}f^n(x_1) - (b-x_1)^{p-1}k = 0$$

$$k = \frac{(b-x_1)^{n-p}}{n-1}f^n(x_1) = \frac{(b-a)^{n-p}(1-\theta)^{n-p}}{(n-1)!}f^n[a+\theta(b-a)]$$

Then

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(b-a)^n(1-\theta)^{n-p}}{p.(n-1)!}f^n[a+\theta(b-a)]$$

The last term assumes a simple form when $p=1$ or n. Then we have

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a)$$

$$+ \frac{(b-a)^n (1-\theta)^{n-1}}{(n-1)!} f^n[a + \theta(b-a)]$$

and

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a)$$

$$+ \frac{(b-a)^n}{n!} f^n[a + \theta(b-a)]$$

Suppose the value of the function and all of its derived functions are known for $x = a$, and that the value of the function of any other x is required, then by putting $x = b$ we have.

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)} f^{n-1}(a) + R$$

where

$$R = \frac{(x-a)^n}{n!} f^n[a + \theta(x-a)] \text{ or } \frac{(x-a)^n (1-\theta)^{n-1}}{(n-1)!} f^n[a + \theta(x-a)]$$

On the right side of the equation all the terms are known except the last in which the unknown θ occurs. If we can prove that the term R formed for arbitrarily increasing values of n,

pass through a series of numbers having the limit zero, then neglecting the last term and summing up as many terms as we please of the infinite series, the following will hold.

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots\cdots$$

This is Taylor's series.

Proof based directly on Rolle's theorem: * Suppose $f(x)$ and its successive $n+1$ differential coefficients are finite and continuous between $x = a$ and $x = a+h$. Let

$$\phi(x) = f(a+x) - f(a) - f'(a)x - \frac{x^2}{2!} f''(a) - \cdots\cdots - \frac{x^n}{n!} f^n(a) - \frac{x^{n+1}}{(n+1)!} R$$

where

$$R = \frac{(n+1)!}{h^{n+1}} \left[f(a+h) - f(a) - f'(a)h - f''(a)\frac{h^2}{2!} - \cdots\cdots - f^n(a)\frac{h^n}{n!} \right]$$

R is independent of x.

$\phi(x) = 0$ when $x = 0$ and when $x = h$.

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* Osborne: Differential and Integral Calculus p 65

Hence by Rolle's theorem $\phi'(x) = 0$ for some value of x between 0 and h. Suppose h' is the value. Then

$$\phi'(x) = f'(a+x) - f'(a) - x f''(a) - \frac{x^2}{2!} f'''(a) - \cdots - \frac{x^{n-1}}{(n-1)!} f^n(a) - \frac{x^n}{n!} R = 0, \quad \text{when } x = h'$$

But $\phi'(x) = 0$ when $x = 0$. Hence $\phi''(x) = 0$ for some value of x between 0 and h'. Continuing this process to $(n+1)$ differentiations, we find that

$$\phi^{n+1}(x) = f^{n+1}(a+x) - R = 0$$

for some value of x between 0 and h. Let this value of x be θh where $0 < \theta < 1$. Then $f^{n+1}(a + \theta h) = R$.

Equating this value of R with that given above, we have

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^n}{n!} f^n(a) + \frac{h^{n+1}}{(n+1)!} f^{n+1}(a + \theta h)$$

Now if we substitute x for a, since a may have any value, we will have

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \cdots + \frac{h^n}{n!} f^n(a) + \frac{h^{n+1}}{(n+1)!} f^{n+1}(a+\theta h)$$

The last term, $\frac{h^{n+1}}{(n+1)!} f^{n+1}(x+\theta h)$ is the remainder after $n+1$ terms. When the form of $f(x)$ is such that by taking n sufficiently large this remainder can be made indefinitely small, then Taylor's theorem gives a convergent series.

*J. H. Miller gives a similar proof but he does not consider that the functions are finite and continuous or the convergency of the series.

Proof based on the generalized law of the mean. §Let $F(x)$ and $f(x)$ be two functions of x, which with their differential coefficients remain finite and continuous between x_1 and x_1+h of the variable x. Suppose between these values $f'(x)$ does not change sign; that is between these values $f(x)$ continually

*Miller: Introduction to Differential and Integral Calculus p 31.

§Todhunter: Differential Calculus; p 76.

increases or decreases. Then

$$\frac{F(x_1+h)-F(x_1)}{f(x_1+h)-f(x_1)} = \frac{F'(x)}{f'(x)}$$

when x has some value between x_1 and x_1+h; that is, $x_1+\theta h$.

$$\therefore \frac{F(x_1+h)-F(x_1)}{f(x_1+h)-f(x_1)} = \frac{F'(x_1+\theta h)}{f'(x_1+\theta h)}$$

Let A and B be the least and greatest values that $\frac{F'(x)}{f'(x)}$ can assume between x_1 and x_1+h. Then $\frac{F'(x)}{f'(x)}-A$ and $\frac{F'(x)}{f'(x)}-B$ will have different signs. The same is the case with $F'(x)-Af'(x)$ and $F'(x)-Bf'(x)$, since $f'(x)$ is of unchangeable sign. These expressions are differential coefficients of $F(x)-Af(x)$ and $F(x)-Bf(x)$. Therefore, of these one must constantly increase and the other decrease. Subtracting their initial from their final values, we have

$$F(x_1+h)-F(x_1)-A[f(x_1+h)-f(x_1)]$$

and $F(x_1+h) - F(x_1) - B[f(x_1+h) - f(x_1)]$.
One will be positive and the other negative.
Hence $\dfrac{F(x_1+h) - F(x_1)}{f(x_1+h) - f(x_1)} - A$
and $\dfrac{F(x_1+h) - F(x_1)}{f(x_1+h) - f(x_1)} - B$
are of contrary signs.
Since $\dfrac{F(x_1+h) - F(x_1)}{f(x_1+h) - f(x_1)}$
is greater than A and less than B,
it must be between the greatest
and least values of $\dfrac{F'(x)}{f'(x)}$

Moreover, this fraction in passing
from the greatest to the least value
must pass through every immediate
value. Hence there must be some
proper fraction θ such that

$$\frac{F(x_1+h) - F(x_1)}{f(x_1+h) - f(x_1)} = \frac{F'(x_1+\theta h)}{f'(x_1+\theta h)}$$

Suppose $f(x) = x - x_1$. Then $f'(x) = 1$.
Since $f(x_1+h) = h$ and $f(x_1) = 0$, then
$F(x_1+h) - F(x_1) = h F'(x_1+\theta h)$

Suppose now that $F(x)$ and $f(x)$ and all of their differential coefficients up to the $(n+1)^{th}$ inclusive are finite and continuous between the values x_1 and $x_1 + h$, of the variable x. Moreover, suppose that one of the two derivatives $F'(x)$ and $f'(x)$ is of invariable sign, and also one of the two derivatives $F''(x)$ and $f''(x)$, and so on to $F^{n+1}(x)$ and $f^{n+1}(x)$.

Then
$$\frac{F(x_1+h) - F(x_1)}{f(x_1+h) - f(x_1)} = \frac{F'(x_1+\theta_1 h)}{f'(x_1+\theta_1 h)}$$

$$\frac{F'(x_1+\theta_1 h) - F'(x_1)}{f'(x_1+\theta_1 h) - f'(x_1)} = \frac{F''(x_1+\theta_2 h)}{f''(x_1+\theta_2 h)}$$

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$$\frac{F^n(x_1+\theta_n h) - F^n(x_1)}{f^n(x_1+\theta_n h) - f^n(x_1)} = \frac{F^{n+1}(x_1+\theta h)}{f^{n+1}(x_1+\theta h)}$$

Suppose that $F'(x), F''(x), \cdots F^n(x), f'(x), f''(x), \cdots f^n(x)$ all vanish when $x = x_1$. Then
$$\frac{F(x_1+h) - F(x_1)}{f(x_1+h) - f(x_1)} = \frac{F^{n+1}(x_1+\theta h)}{f^{n+1}(x_1+\theta h)}$$

If we take $f(x) = (x - x_1)^{n+1}$, all the

conditions are satisfied.

$f^{n+1}(x) = (n+1)!$ $\quad f(x_1) = 0$ $\quad f(x_1+h) = h^{n+1}$

Suppose then that $F(x)$ and all the differential coefficients are continuous between x_1 and x_1+h of x, and all the differential coefficients up to the n^{th} inclusive vanish when $x = x_1$. Then

$$F(x_1+h) - F(x_1) = \frac{h^{n+1}}{(n+1)!} F^{n+1}(x_1 + \theta h)$$

Suppose $x_1 = 0$, then $F(x_1) = 0$ Then

$$F(h) = \frac{h^{n+1}}{(n+1)!} F^{n+1}(\theta h) \qquad (1)$$

Let $\phi(x+h)$ be a function to be expanded in a series of ascending powers of h. Let

$$\phi(x+h) - \phi(x) - h\phi'(x) - \frac{h^2}{2!}\phi''(x) - \cdots\cdots - \frac{h^n}{n!}\phi^n(x) = F(h)$$

Then $F(h)$ and the differential coefficients with respect to h up to the n^{th} inclusive vanish when $h=0$. Also

$$F^{n+1}(h) = \phi^{n+1}(x+h)$$

Hence by (1), we have

$$F(h) = \frac{h^{n+1}}{(n+1)!} F^{n+1}(\theta h) = \frac{h^{n+1}}{(n+1)!} \phi^{n+1}(x+\theta h)$$

$$\therefore \phi(x+h) = \phi(x) + h\phi'(x) + \frac{h^2}{2!}\phi''(x) + \cdots\cdots$$

$$+ \frac{h^n}{n!}\phi^n(x) + \frac{h^{n+1}}{(n+1)!}\phi^{n+1}(x+\theta h).$$

From this, Taylor's theorem, it follows that whenever the function is such that by sufficiently increasing n, $\frac{h^{n+1}}{(n+1)!}\phi^{n+1}(x+\theta h)$ can be made as small as we please, and the series converges.

Proof based on the binomial theorem.* When $f(x)$ is a polynomial of the n^{th} degree, we see from algebra that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots\cdots + \frac{h^n}{n!}f^n(a) \quad (1)$$

and the development stops of itself because all the derivatives after the $(n+1)^{th}$ are zero. If we wish to apply

*Goursat: Cours D'analyse Mathématique p 101

this formula is a function different from a polynomial, the right hand member becomes an infinite series. Suppose $f(x)$ and its first n derivatives are finite and continuous for x between a and $a+h$. Having given a and h, we place

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(x) + \cdots + \frac{h^n}{n!} f^n(a) + \frac{h^p}{p} P, \qquad (2)$$

p being some positive number.

Place

$$\phi(x) = f(a+h) - f(x) - (a+h-x) f'(x) - \frac{(a+h-x)^2}{2!} f''(x) - \cdots - \frac{(a+h-x)^n}{n!} f^n(x) - \frac{(a+h-x)^p}{p!} P.$$

and we have $\phi(a) = 0$ and $\phi(a+h) = 0$ The first relation results from substituting the value of P of (2) in the equation for $\phi(x)$.

Since $\phi(a)$ and $\phi(a+h)$ both equal zero, then by Rolle's theorem, there is some value of x, say $a+\theta h$, in the interval

$(a, a+h)$ for which $\phi'(x) = 0$ Then

$$\phi'(x) = \frac{(a+h-x)^{p-1}}{n!}\left[F - (a+h-x)^{n-p+1} f^{n+1}(x)\right]$$

since $\phi'(x) = 0$ for $x = a + \theta h$, then

$$P = h^{n-p+1}(1-\theta)^{n-p+1} f^{n+1}(a+\theta h) \text{ for } 0 < \theta < 1.$$

Replacing this value of P in (2), we have

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots$$

$$+ \frac{h^n}{n!} f^n(a) + R_n \qquad (3)$$

where $R_n = \frac{h^{n+1}(1-\theta)^{n-p+1}}{p!} f^{n+1}(a+\theta h)$

If n becomes infinitely large, $R_n \doteq 0$, and the series converges.

*Laurent says nothing about the convergence of the series, and §Genocchi says nothing about the function and its derivatives being finite and continuous.

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*Laurent: Traité D'Analyse Vol I. p 75

§Genocchi-Peano: Differential und Integral Rechnung p 68

Proof of Maclaurin's Theorem

*Let $f(x)$ be a function of x, which together with its $(n-1)$ derivatives is continuous for values of x from 0 to h including 0 and h. Let us write

$$f(x) = \phi_n(x) + R_n(x) \quad (1)$$ where

$$\phi_n(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0), \quad (2)$$

that is $\phi_n(x)$ is the sum of the first n terms of Maclaurin's expansion, and $R_n(x)$ is a symbol for the difference between $f(x)$ and $\phi_n(x)$. If we can show that by taking n large enough, a point can be reached after which the values of $R_n(x)$ will be less than any assigned magnitude, then Maclaurin's series is convergent, and its sum to infinity is $f(x)$. It is evident that the argument can not be pushed to this conclusion, if $f(x)$ or any of its derivatives are discontinuous for any value of x belonging

*Lamb: Infinitesimal Calculus p. 576.

to the range considered.

From the conditions that $\phi_n(x)$ has been made to satisfy, it appears that $R_n(x)$ and its first $(n-1)$ derivatives will be continuous from $x=0$ to $x=h$ and will all vanish for $x=0$.

Let $F(x)$ be the same sort of function as $R_n(x)$ and let A and B be the lower and upper limits to the values which the n^{th} derivative assumes in the interval $(0, h)$. We have

$$F(0)=0,\ F'(0)=0,\ F''(0)=0 \text{ ------ } F^{n-1}(0)=0$$

and $$A < F^n(x) < B$$

The latter condition holding from $x=0$ to $x=h$. We have

$$\int_0^x A\,dx < \int_0^x F^n(x)\,dx < \int_0^x B\,dx,$$

or since $F^{n-1}(0)=0$

$$Ax < F^{n-1}(x) < Bx,$$

and we have

$$\int_0^x Ax\,dx < \int_0^x F^{n-1}(x)\,dx < \int_0^x Bx\,dx.$$

Since $F^{n-2}(0)=0$,

$$A\frac{x^2}{2!} < F^{n-2}(x) < B\frac{x^2}{2!}$$

also $A\frac{x^3}{3!} < F^{n-3}(x) < B\frac{x^3}{3!}$

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$$A\frac{x^n}{n!} < F(x) < B\frac{x^n}{n!}$$

Hence, we may write $F(x) = b\frac{x^n}{n!}$, where b is some quantity between A and B. Since $\phi_n(x)$ is a rational integral function of the degree $(n-1)$, its n^{th} derivative is zero; and the n^{th} derivative of $R_n(x)$ is $f(1)$ equal to $f^n(x)$. We infer then that

$$R^n_n(x) = b\frac{x^n}{n!},$$

If $f^n(x)$, as we shall suppose, this derivative, is continuous from $x=0$ to $x=h$, there will be some value between 0 and h for which $f^n(x) = b$. Denote this value by θh, and we have

$$R_n(x) = \frac{x^n}{n!} f^n(\theta h) \quad \text{for } 0<\theta<1.$$

This is true from $x=0$ to $x=h$, inclusive. Put $x=h$ and substitute in (1), and we obtain

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(0) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(h) + \frac{h^n}{n!}f^n(\theta h)$$

If we write $f(x) = \phi(a+x)$, we deduce

$$\phi(a+h) = \phi(a) + h\,\phi'(a) + \frac{h^2}{2!}\phi''(a) + \cdots\cdots$$
$$+ \frac{h^{n-1}}{(n-1)!}\phi^{n-1}(a) + \frac{h^n}{n!}f^n(a+\theta h) \text{ for } 0<\theta<1$$

This is Taylor's theorem, and holds on the assumption that $\phi^n(x)$ exists, and is a continuous from $x=a$ to $x=a+h$ inclusive.

In this proof, Lamb says nothing about the convergence of the series.

Picard's proof.* Integrating $u\frac{dv}{dx}dx$ by parts we have

$$\int_a^b u\frac{dv}{dx}dx = (uv)_a^b - \int_a^b v\frac{du}{dx}dx,$$

where $(uv)_a^b$ represents the difference of value of the product for $x=a$ and $x=b$. This integration may, in general, be carried on n times.

Let us take $f(x)$ and $y(x)$ and form the identity

*Picard: Traité d'analyse Vol 1, p. 13.

$$f^{(h)}(x)\, g^{(n-h)}(-x) - f^{(h-1)}(x)\, g^{(n-h+1)}(-x) = \frac{d}{dx}\left[f^{(h-1)}(x)\, g^{(n-h)}(-x)\right]$$

By placing successively $h = 1, 2, \ldots n$, and adding, we have

$$f^{(n)}(x)\, g(-x) - f(x)\, g^{(n)}(-x) = \sum_1^n h \frac{d}{dx}\left[f^{(h-1)}(x)\, g^{(n-h)}(-x)\right].$$

Integrating between a and b, we have

$$\int_a^b f^{(n)}(x)\, g(-x)\, dx - \int_a^b f(x)\, g^{(n)}(-x)\, dx =$$

$$\sum_1^n h \left[f^{(h-1)}(x)\, g^{(n-h)}(-x)\right]_a^b$$

Place $f(x) = F'(x)$, $g(x) = \frac{(x+b)^n}{n!}$,

and we have

$$\int_a^b F^{(n+1)}(x) \frac{(b-x)^n}{n!}\, dx - \int_a^b F'(x)\, dx =$$

$$\sum_1^n h \left[F^{(h)}(x) \frac{(b-x)^h}{h!}\right]_a^b = \sum_1^n h\, F^{(h)}(a) \frac{(b-a)^h}{n!}$$

Expanding we have

$$F(b) = F(a) + (b-a)F'(a) + \cdots + \frac{(b-a)^n}{n!} F^{(n)}(a)$$

$$+ \int_a^b F^{(n+1)}(x) \frac{(b-x)^n}{n!}\, dx$$

This is Taylor's series with the

remainder in the form of a definite integral. We may put the remainder in another form as follows:

$$\int_a^b F^{n+1}(x)\frac{(b-x)^n}{n!}dx = F^{n+1}(\xi)\int_a^b \frac{(b-x)^n}{n!}dx$$

$$= F^{n+1}(\xi)\frac{(b-a)^{n+1}}{(n+1)!}$$

where $a < \xi < b$.

Picard does not consider that the series must converge, or that the function and its derivatives must be finite and continuous in a given interval.

Jordan's proof.* Let $f(x)$ be a real function of a real variable x, which together with its first n derivatives is continuous in the interval AA'. Denote the two extremities of this interval by a and $a+h$ and place

$$(1)\quad f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n$$

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*Jordan: Cours D'analyse Vol 1, p 245

The remainder R_n is a new function of h which we are to determine. In taking the successive derivatives of R_n with respect to h, we see that the n^{th} derivative of R_n equals $f^n(a+h)$, and R_n and its $(n-1)$ derivatives vanish for $h=0$. These two conditions are enough to completely define R_n, because if ϕ is a function which is satisfied, the difference $R_n - \phi$, having its n^{th} derivative equal to zero, would be a polynomial of the $(n-1)^{th}$ order; but this polynomial and its $(n-1)$ derivatives vanish for $h=0$.

We may see that the definite integral

$$I = \frac{1}{(n-1)!}\int_a^{a+h} (a+h-x)^{n-1} f^n(x)\,dx$$

satisfies these conditions.

In order to show this, let $x = a+t$.

$$I = \frac{1}{(n-1)!}\int_0^h (h-t)^{n-1} f^n(a+t)\,dt.$$

$$\frac{dI}{dh} = \frac{1}{(n-2)!}\int_0^h (h-t)^{n-2} f^n(a+t)\,dt$$

$$\frac{d^2I}{dh^2} = \frac{1}{(n-3)!}\int_0^h (h-t)^{n-3} f^n(a+t)\,dt$$

$$\frac{d^{n-1}I}{dh^{n-1}} = \int_0^h f^n(a+t)\,dt.$$

All of these vanish for $h=0$; finally by the last differentiation, we get

$$\frac{d^nI}{dh^n} = f^n(a+h)$$

We get for R_n the following expression:

$$R_n = \frac{1}{(n-1)!}\int_0^h (h-t)^{n-1} f^n(a+t)\,dt.$$

Let $t = uh$, where $0 < u < 1$. We have

$$R_n = \frac{h^n}{(n-1)!}\int_0^1 (1-u)^{n-1} f^n(a+uh)\,du \quad (2)$$

Let p be an arbitrary function not greater than n. The function to be integrated is the product of two factors $(1-u)^{p-1}$ and $(1-u)^{n-p} f^n(a+uh)$, of which the

first is positive and the second is continuous in the interval of integration. We have then on applying the theorem of integration by parts

$$R_n = \frac{h^n (1-\theta)^{n-p} f^n(a+\theta h)}{(n-1)!} \int_0^1 (1-u)^{p-1} du$$

where $0 < \theta < 1$. Integrating again

$$\int_0^1 (1-u)^{p-1} du = \left[-\frac{(1-u)^p}{p} \right]_0^1 = \frac{1}{p}.$$

Then

$$R_n = \frac{(1-\theta)^{n-p} f^n(a+\theta h)}{(n-1)!\, p}.$$

and if $p = n$

$$R_n = \frac{h^n}{n!} f^n(a+\theta h) \qquad (3)$$

If n is constant and h becomes infinitely small, then (2) and (3) show that R_n becomes infinitely small. We have for infinitely small $f(a+h) - f(a)$, the value

$$h f'(a) + \frac{h^2}{2!} f''(a) + \text{------} + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a)$$

approaches, indefinitely, near to the order in question, on the other hand, that it is constant or stays within certain limits, and that it increases indefinitely. In this case R_n approaches zero and

$$f(a+h) = \lim\left[f(a) + h f'(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a)\right]$$

In this case, the infinite series

$$f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots$$

converges toward $f(a+h)$.

Since the value of the function of a and all its derivatives are known, and we are required to find the value of any x, we will have the series

$$f(x) + h f'(x) + \frac{h^2}{2!} f''(a) + \cdots$$

converging toward $f(x+h)$.

Jordan does not consider that $f(x)$, $f'(x)$, $f''(x)$ --- must be finite and continuous in a given interval.

Walton's proof.* Whatever be the function, provided it remains continuous between the values of the variable denoted by x and $x+h$, we shall have

$$f(x+h) - f(x) = \int_x^{x+h} f'(x)\, dx.$$

In order to express more clearly the distinction between the general value of x under the sign of integration, and the value of x which enters into the expression of the limit, we write

$$f(x+h) - f(x) = \int_x^{x+h} f'(x')\, dx'$$

Putting $x' = x + h - z$ and $dx' = -dz$, we have

$$f(x+h) - f(x) = -\int_h^0 f'(x+h-z)\, dz$$

$$= \int_0^h f'(x+h-z)\, dz$$

Integrating by parts

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*Walton: Differential Calculus p. 133

$$\int_0^h f'(x+h-z)\,dz = h f'(x) + \int_0^h z f''(x+h-z)\,dz$$

Hence

$$f(x+h) = f(x) + h f'(x) + \int_0^h z f''(x+h-z)\,dz$$

Integrating again by parts

$$\int_0^h z f''(x+h-z)\,dz = \frac{1}{2}h^2 f''(x) + \frac{1}{2}\int_0^h z^2 f'''(x+h-z)\,dz$$

Hence

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{1}{2}\int_0^h z^2 f'''(x+h-z)\,dz.$$

In general by a series of successive integration, we have

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots\cdots + \frac{h^\nu}{\nu!} f^\nu(x) + \frac{1}{\nu!}\int_0^h z^\nu f^{\nu+1}(x+h-z)\,dz.$$

Suppose that P and Q represent the least and greatest values of $f^{\nu+1}(x+h-z)$ for a series of values of z between 0 and h; it is evident we shall have

$$\left[P\int_0^h z^r dz = \frac{Ph^{r+1}}{r+1}\right] < \int_0^h z^r f^{r+1}(x+h-z)\,dz <$$

$$\left[Q.\int_0^h z^r dz = \frac{Q\,h^{r+1}}{r+1}\right]$$

It appears that the value of R_r falls between $\frac{Ph^{r+1}}{(r+1)!}$ and $\frac{Q\,h^{r+1}}{(r+1)!}$

and that moreover

$$R_r = \frac{1}{r!}\int_0^h z^r f^{r+1}(x+h-z)\,dz$$

which is the value of the remainder after a finite number of terms.

The convergency of the series was not considered by Walton.

Miscellaneous Notes. Walton gives a proof which is practically the same as the original proof that Taylor gave. However he develops a form of the remainder, and considers the convergency of the infinite series. He says that the original function and derivative

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* Walton: Differential Calculus p 128

must be finite but says nothing of continuity.

Hemming's proof. *If $\phi(x)$ is any function of x such that $\phi(a) = 0$, and $\phi'(x)$ is positive for all values of x from a to $a+h$ inclusive, then $\phi(x)$ will have the same sign as h; and if $\phi'(x)$ is negative for all values within these limits, then $\phi(x)$ will have a different sign from h.

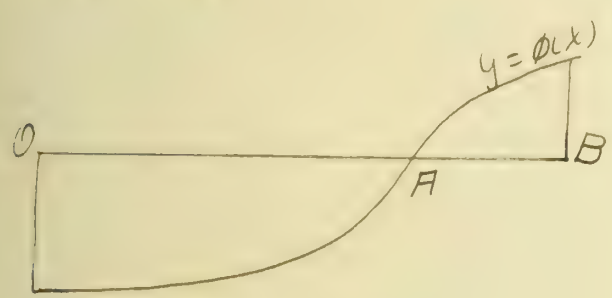

Fig I

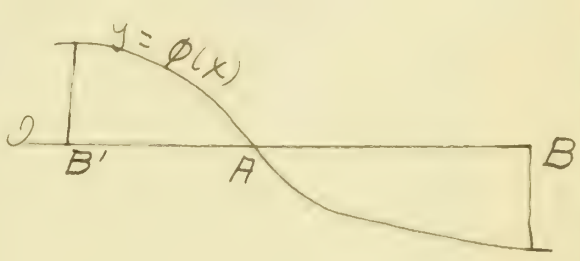

Fig II

In order to prove the above statement, let OAB be the axis of x, OA being equal to a and OB being equal to $a+h$ where h is positive. Then the curve $y = \phi(x)$ must by the assumed condition pass through A, and (if the

*Hemming; Differential and Integral Calculus p 104.

differential coefficient is positive from A to E) the curve will be of the form of that in FIS I and will be positive between A and E. If h is negative and OE' less than OA, y will be negative. If ϕ' is negative, the curve will be as in FIS II and will have a sign different from that of h.

Let $F(x)$ be a function of x such that $F(a) = 0$ and let A and E be the greatest and least values of

$$\frac{F'(x)}{n(x-a)^{n-1}},$$

when x varies from a to $a+h$. Then between these limits

$$A - \frac{F'(x)}{n(x-a)^{n-1}} \quad \text{and} \quad \frac{F'(x)}{n(x-a)^{n-1}} - E$$

are constantly positive.

$\therefore A - n(x-a)^{n-1} - F'(x)$ and $F'(x) - E n(x-a)^{n-1}$ are constantly positive or constantly negative, since $(x-a)^{n-1}$ can not change its sign within the given limits.

If $\phi_1(x) = A(x-a)^n - F(x)$,
and $\phi_2(x) = F(x) - B(x-a)^n$,
the above quantities will be the values of $\phi_1'(x)$ and $\phi_2'(x)$, and $\phi_1(x)$ and $\phi_2(x)$ will satisfy the conditions of the lemma given at the first of the proof, and $\phi_1(x)$ and $\phi_2(x)$ will both be positive or both negative within the given limits.

$\therefore A - \frac{F(x)}{(x-a)^n}$ and $\frac{F(x)}{(x-a)^n} - B$

have the same sign; by giving to x one of the admissible values viz; $a+h$, then $\frac{F(a+h)}{h^n}$ lies between A and B. This may be expressed in the equation:

$$\frac{F(a+h)}{h^n} = \frac{F'(a+\theta_1 h)}{n(\theta_1 h)^{n-1}}$$

where θ_1 has some value between 0 and 1, since the quantity on the right of the equation may, by varying θ_1 from 0 to 1, be made to assume all

the values from E to A, some one of which must satisfy the above equation. If not only $F(a)$ but also $F'(a)$ equals zero, we may prove in the same way that

$$\frac{F'(a+\theta_1 h)}{(\theta_1 h)^{n-1}} = \frac{F''(a+\theta_2 h)}{(n-1)(\theta_2 h)^{n-2}},$$

where θ_2 has some value between 0 and θ_1. If all the quantities $F(a), F'(a), F''(a) \ldots F^{n-1}(a)$ equal zero, we may obtain a series of equations similar to the above, of which the last is

$$\frac{F^{n-1}(a+\theta_{n-1} h)}{(\theta_{n-1} h)} = \frac{F^n(a+\theta h)}{1},$$

where θ must lie between 0 and 1. Collecting these equations, we have

$$\frac{F(a+h)}{h^n} = \frac{F^n(a+\theta h)}{n!} \qquad (1)$$

We will now give some particular value to $F(x)$.

$$F(x) = f(x) - \left[f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \ldots + f^{n-1}(a)\frac{(x-a)^{n-1}}{(n-1)!}\right]$$

By differentiating $F(x)$, we have

$$F'(x) = f'(x) - \left[f'(a) - f''(a)(x-a) + \cdots + f^{n-1}(a)\frac{(x-a)^{n-2}}{(n-2)!} \right]$$

$$F''(x) = f''(x) - \left[f''(a) + f'''(a)(x-a) + \cdots + f^{n-1}(a)\frac{(x-a)^{n-3}}{(n-3)!} \right]$$

$$F^r(x) = f^r(x) - \left[f^r(a) + f^{r+1}(a)(x-a) + \cdots + f^{n-1}(a)\frac{(x-a)^{n-r+1}}{(n-r+1)!} \right]$$

$$F^{n-1}(x) = f^{n-1}(x) - f^{n-1}(a)$$

$$F^n(x) = f^n(x).$$

All the terms within the brackets, except the first, contain some power of $(x-a)$, and will therefore vanish when $x=a$, unless some of their coefficients are infinite. If therefore $f(x)$ be such a function that none of the differential coefficients $f'(a)$, $f''(a)$, $f'''(a)$ $\cdots\cdots$ $f^{n-1}(a)$ are infinite, we shall have

$$F(a)=0,\ F'(a)=0,\ \cdots\cdots F^r(a)=0 \cdots\cdots F^{n-1}(a)=0,\ F^n(x)=f^n(x)$$

Therefore $F(x)$ satisfies the conditions necessary to equation (1), and by substituting the values of $F(x)$ and $F^n(x)$

in that equation, it becomes

$$f(a+h) = f(a) + f'(a)\,h + f''(a)\,\frac{h^2}{2!} + \cdots\cdots$$

$$+ f^{n-1}(a)\,\frac{h^{n-1}}{(n-1)!} + f^{n}(a+\theta h)\,\frac{h^n}{n!} \qquad (2)$$

In this expression we may give to n any value we please; and if none of the differential coefficients of $f(x)$ become infinite when $x = a$, n may be taken indefinitely large and we have

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots$$

The series in this form is Taylor's series.

* Lamb says nothing about the convergence of the series, or that $f(x)$ and its derivatives must be continuous.

Lamb's proof. * Consider any given curve $y = f(x)$ (1) and the curve

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*Lamb: Infinitesimal Calculus p 582

$$y = A_0 + A_1 x + A_2 x^2 + \cdots + A_{n-1} x^{n-1} + A_n x^n, \quad (2)$$

in which the $(n+1)$ coefficients are determined so as to make the curves intersect at $x = 0$ and $x = h$, and so as to make the values of

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots \cdots \frac{d^{n-1}y}{dx^{n-1}}$$

respectively the same in the two curves at $x = 0$.

These conditions give

$$A_0 = f(0), A_1 = f'(0), A_2 = \frac{f''(0)}{2!} \cdots A_{n-1} = \frac{f^{n-1}(0)}{(n-1)!} \quad (3)$$

and

$$f(h) = A_0 + A_1 h + A_2 h^2 + \cdots + A_{n-1} h^{n-1} + A_n h^n \quad (4)$$

Denoting the difference of the ordinates of the two curves by $F(x)$, we have:

$$F(0) = 0, F'(0) = 0, F''(0) = 0, \cdots F^{n-1}(0) = 0 \quad (5)$$

and $$F(h) = 0 \quad (6)$$

Since $F(x)$ vanishes for $x = 0$ and $x = h$, it follows by Rolle's theorem that $F'(x)$ vanishes for some value of

of x between 0 and h, say for $x = \theta_1 h$, where $0 < \theta < 1$. Since $F'(x)$ vanishes for $x = 0$ and $x = \theta_1 h$, $F''(x)$ will vanish for some value of x between $x = 0$ and $x = \theta_1 h$, say for $x = \theta_2 h$, where $0 < \theta_2 < \theta_1$. Proceeding in this way, we find that $F^{n-1}(x)$ vanishes for $x = 0$ and $x = \theta_{n-1} h$ where $1 > \theta_{n-1} > 0$. Hence

$$F^n(\theta h) = 0, \qquad (7)$$

where $1 > \theta > 0$. By referring to (1) and (2), we see that

$$F^n(x) = f^n(x) - n! A_n \qquad (8)$$

From (7)

$$A_n = \frac{1}{n!} f^n(\theta h) \qquad (9)$$

Substituting from (3) and (9) in (4), we have

$$f(h) = f(a) + h f'(0) + \frac{h^2}{2!} f''(0) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{h^n}{n!} f^n(\theta h) \qquad (10)$$

Let $f(x) = \phi(a+x)$. Then we have

$$\phi(a+h) = \phi(a) + h\phi'(a) + \frac{h^2}{2!}\phi''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}\phi^{n-1}(a) + \frac{h^n}{n!}\phi^n(a+\theta h) \qquad (11)$$

In this proof Lamb does not bring in the subject of continuity and convergence of the series. The proof restricts the application of the theorem to cases where the function can be represented by curves, and the continuity of these curves is not considered.

Stolz's proof. *Let the given integral function of x be

$$F(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

Then we can change $F(x+\xi)$ into an integral function of ξ by the help of the binomial theorem.

$$(x+\xi)^m = x^m + m x^{m-1}\xi + \frac{m}{2} x^{m-2}\xi^2 + \cdots + \xi^m \qquad (1)$$

We can then write

$$F(x+\xi) = F(x) + F'(x)\xi + \frac{F''(x)}{2!}\xi^2 + \cdots + \frac{F^n(x)}{n!}\xi^n$$

We can show that

*Stolz: Differential Rechnung Vol 1, p 15

$$F'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}$$

$$F''(x) = 2.1\,a_2 + 3.2\,a_3x + \cdots + n(n-1)a_nx^{n-2}$$

The above process of expanding $F(x+\xi)$ according to powers of ξ, is a special case of that to be considered next. Suppose we have the infinite series:

$$\sum_0^\infty a_n(x-a)^n, \qquad (2)$$

in which a may be zero. This series converges absolutely for all values of $(x-a)$, the absolute value of which is smaller than a given positive value R *

The sum of series for all values of x within the interval $(a-R, a+R)$

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* If a power series, $P(x)$, is convergent when $x = x_0$, it is absolutely convergent when $|x| < |x_0|$.

Harkness and Morley: Introduction to Analytic Functions. Chap. X §76.

may be expressed by $f(x)$, so that $f(a) = a_0$.

If x denote a value for which $|x-a| < R$, then by a suitable limitation of ξ, we can change $f(x+\xi)$ into a power series of ξ, so that $[(x-a)+\xi]^m$ may be developed in the form of equation (1). Then we have the following theorem:

"If $|x-a| < R$, then the infinite series

$$\sum_m^\infty n(n-1)\cdots(n-m+1)\,a_n(x-a)^{n-m} \qquad (3)$$

for $m = 1, 2, 3 \ldots\ldots$

converges absolutely. If we denote their sums by $f'(x)$, $f''(x) \ldots\ldots f^m(x) \ldots\ldots$, then for all values of ξ which fulfill the condition

$$|x-a| + |\xi| < R$$

the series

$$f(x) + \sum_1^\infty m \frac{f^m(x)}{m!}\xi^m \qquad (4)$$

converges absolutely, and $f(x+\xi)$ is their

sum."

By differentiating the power series (2) we obtain an unlimited number of new power series of x. These are the series

$$f'(x) = \sum_{1}^{\infty n} n a_n (x-a)^{n-1} \qquad (5)$$

$$f''(x) = \sum_{2}^{\infty n} n(n-1) a_n (x-a)^{n-2} \qquad (6)$$

$$f^{(m)}(x) = \sum_{m}^{\infty n} n(n-1)\cdots(n-m+1) a_n (x-a)^{n-m} \qquad (7)$$

All of these functions are defined for all values of x in the interval $(a-R, a+R)$. Then we have

$$f'(a) = a_1, \quad f''(a) = 2a_2, \quad \cdots \quad f^{n}(a) = n! a_n \cdots \qquad (8)$$

If we start with a definite single valued function $f(x)$, then we have for some definite value of x, say $x=a$, derivatives only when there is a certain positive number R, so that $f(x)$ for all values of x in the interval $(a-R, a+R)$ is represented by a converging power series (2). According

To (7) we may put the series (2) in the following form:

$$f(x) = f(a) + \sum_{1}^{\infty m} \cdot \frac{f^m(a)}{m!}(x-a)^m,$$

which is the form of Taylor's series.

*Stolz also puts Taylor's series in a form which has a remainder.

We must consider the difference

$$f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 - \cdots\cdots - \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1} \qquad (9)$$

Let x be considered as fixed, say $x = b$, and denote the function by D. We then have

$$f(b) - f(a) - f'(a)(b-a) - \frac{f''(a)}{2!}(b-a)^2 - \cdots\cdots - \frac{f^{n-1}(a)}{(n-1)!}(b-a)^{n-1} = D. \qquad (10)$$

Place $a = y$, and define the function $F(y)$ by the formula.

$$F(y) = f(y) + f'(y)(b-y) + \frac{f''(y)}{2!}(b-y)^2 + \cdots + \frac{f^{n-1}(y)}{(n-1)!}(b-y)^{n-1} \qquad (11)$$

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* Stolz: Differential Rechnung Vol 1, p. 93.

Then $D = f(b) - F(a)$.

Differentiating (11), we have

$$F'(y) = \frac{(b-y)^{n-1}}{(n-1)!} f^n(y).$$

By the general law of the mean we have

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(X)}{\phi'(X)} \quad \text{for } a < X < b.$$

Substituting the value D for $f(b)-f(a)$ and the value of $F'(y)$ for $f'(X)$, we have

$$\frac{D}{\phi(b)-\phi(a)} = \frac{(b-y)^{n-1}}{(n-1)!} \frac{f^n(y)}{\phi'(y)} \qquad (12)$$

In (12) y is an unknown value between a and b, and can be given a value $[a+\theta(b-a)]$, in which θ is an unknown proper fraction. We have:

$$D = \phi(b)-\phi(a)\left[\frac{\{b-a-\theta(b-a)\}^{n-1}}{(n-1)!}\right]\frac{f^n(a+\theta h)}{\phi'(a+\theta h}$$

$$= \phi(b)-\phi(a)\ \frac{[(b-a)(1-\theta)]^{n-1}}{(n-1)!}\ \frac{f^n(a+\theta h)}{f'(a+\theta h)}$$

$$= \frac{\phi(b)-\phi(a)}{\phi'(a+\theta h)}\ \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h) \qquad (13)$$

where $0 < \theta < 1$ and h is substituted for $(b-a)$.

We had $D = f(b) - F(a)$ (14)

Substituting (13) in (14), we have

$$f(b) = f(a) + \sum_{1}^{n-1} \frac{f^n(a)}{n!}(b-a)^n +$$

$$\frac{\phi(b) - \phi(a)}{\phi'(a+\theta h)} \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h)$$

This is the general form of Taylor's series.

If $\phi(x) = [\epsilon(b-x)]^\nu$,

when $\epsilon = \pm 1$, $(b-a)$ is a positive or negative number, and ν is a positive number, then

$$\phi(b) = 0$$

$$\phi(a) = [\epsilon(b-a)]^\nu$$

$$\phi'(x) = -\nu\epsilon[\epsilon(b-x)]^{\nu-1}$$

$\phi'(x)$ is such a value that it does not vanish or become infinite for any value of x between a and b. Then

$$\frac{\phi(b) - \phi(a)}{\phi'(a+\theta h)} = \frac{h}{\nu(1-\theta)^{\nu-1}}$$

and

$$f(x) = f(a) + \sum_{1}^{n-1} \frac{f^r(a)}{r!}(\delta - a)^r + \frac{h^n(1-\theta)^{n-v}}{(n-1)!\,v} f^n(a + \theta h)$$

for $0 < \theta < 1$ and $v > 0$.

This is a special form of Taylor's series.

Chapter II

Conditions for the existence of Taylor's Theorem.

§ 1. Examples of functions which can not be expanded at given points by Taylor's Theorem. In the preceding chapter we have considered the different ways in which Taylor's theorem may be developed. In this chapter we are to consider a few special cases in which Taylor's theorem does not hold, and also a necessary and sufficient condition for the existence of this theorem.

*In order to expand the function

* Price: Treatise on Infinitesimal Calculus. Vol. I, p. 211.

$F(x+h)$ to $(n+1)$ terms, so that the last term of the series may be the sum of all the terms after the n^{th} term and be

$$F^n(x+\theta h)\frac{h^n}{1.2.3\cdots n},$$

it is necessary that $F(x)$ and all of its derived functions up to $F^n(x)$ inclusively, should not be infinite for any value of x between x and $x+h$.

By differentiating a function, we see that all rational and integral functions of x, and all functions of the form $x^n + cx^{-m} + ex^{p/q}$ are lowered one dimension. If n be positive and integral, the n^{th} derived function of x^n is a constant, and the $(n+1)^{th}$ is zero. However many times x^{-m} may be differentiated, the result is never zero, but the negative exponent is always increased. If p/q is a fraction, proper or improper, some

derived function of x^n will sometimes have a negative exponent, and therefore some power of x will appear in the denominator.

Let a be a particular value of x between x and $x+h$, and suppose that when $x=a$, $F^n(x)$ is finite, but $F^{n+1}(x)$ is infinite. Then it is plain that some factor of the form $(x-a)$ must, in passing from $F^n(x)$ to $F^{n+1}(x)$, have been introduced into the denominator, and that there must have been in the original function a factor of the form $(x-a)^{m+\frac{p}{q}}$, where m is an integral, positive number and $\frac{p}{q}$ is a proper fraction. If the original function and all of its derived functions are infinite, when $x=a$, there must have been in the original function a factor of the form $(x-a)^{-m}$.

In the latter case Taylor's theorem is not applicable at all for values which include $x = a$; and in the former case, suppose we have to expand

$$F(x) = f(x) + (x-a)^{m+\frac{h}{q}} \phi(x).$$

in which neither $F(x)$ nor $\phi(x)$ contain factors of the form $(x-a)$, then all the derived functions up to $F^m(x)$ will be finite when $x = a$, but the subsequent ones will be infinite. The expansion must be carried beyond the m^{th} term, but the addition of

$$\frac{h^m}{1.2.3 \cdots m} F^m(x + \theta h)$$

will make the equation exact.

Example 1.*

Suppose that it is required to expand $F(a+h)$ having given

$$F(x) = x^4 + (x-a)^{\frac{5}{2}} \text{ in } x.$$

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* Price: Treatise on Infinitesimal Calculus Vol. 1, p. 202.

Then a is the least value of x, for which the function is considered, and $a+h$ is the greatest. We have

$$F'(x) = 4x^3 + \tfrac{3}{2}(x-a)^{3/2} \sin x + (x-a)^{3/2} \cos x.$$

$$F''(x) = 12x^2 + \tfrac{15}{4}(x-a)^{1/2} \sin x + 5(x-a)^{3/2} \cos x - (x-a)^{5/2} \sin x.$$

If we form $F'''(x)$, it involves $(x-a)^{-1/2}$, which becomes infinite when $x=a$, and therefore fails to fulfill the conditions for the existence of Taylor's theorem. In this case

$$F(x+h) = F(x) + hF'(x) + \frac{h^2}{2!} F''(x+\theta h),$$

and substituting the values given above, and putting $x=a$, we have

$$F(a+h) = a^4 + 4a^3 h + \frac{h^2}{2!}\left\{12(a+\theta h)^2 + \frac{15}{4}(\theta h)^{1/2} \sin(a+\theta h) + 5(\theta h)^{3/2} \cos(a+\theta h) - (\theta h)^{5/2} \sin(a+\theta h)\right\},$$

θ being a positive, proper fraction. The first three terms of the series are correct terms of the expansion

of $(a+h)^4$, and the last three terms contain h with fractional exponent, giving an illustration of the reason why Taylor's series fails; viz, because we are attempting to expand by a formula which involves only integral powers of h, a function which in its development requires fractional powers.

Let us consider the case of $(x^2-a^2)^{1/2}$, which is to be developed, when $x=a+h$, so that the original function equals zero when $x=a$, and the first derived function equals ∞: as the function may be written in the form $(x-a)^{1/2}(x+a)^{1/2}$, this becomes when $x=a+h$, $h^{1/2}(2a+h)^{1/2}$ the second factor of which may be developed according to Taylor's theorem. Therefore the whole development will consist of terms of the power of h whose exponents are

of the form $n, n+1/2$, n being a whole number.

Similarly, if the original function has factors of the form $(x-a)^{-m}$, then it and all its derived functions are infinite when $x=a$. Suppose when $x=a+h$, we have to develope by Taylor's series the function

$$F(x) = \frac{\sin x}{x^2 - a^2}$$

which with all of its derived functions become infinite when $x=a$. However, the function may be written in the form

$$\frac{1}{x-a} \cdot \frac{\sin x}{x+a}$$

the first factor of which becomes $1/h$ when $x=a+h$, and the second fulfills the conditions of Taylor's series and may be expanded in the ordinary way. The resulting development will have at least one term involving the negative power

of h, which indicates the cause of the failure; viz. that the function does not admit of the development in ascending, integral, positive powers of h, which alone is given by Taylor's theorem.

Example 2.

*Let

$$f(x) = 2ax - x^2 - a\sqrt{x^2 - a^2}$$

$$f'(x) = 2(a - x) + \frac{ax}{\sqrt{x^2 - a^2}}$$

$$f''(x) = -2 + \frac{a}{\sqrt{x^2 - a^2}} - \frac{ax^2}{(x^2 - a^2)^{3/2}}.$$

By placing $x = a$, we have $f(x) = a^2$, and all the differential coefficients become infinite. Therefore the function can not be developed by Taylor's series, as the differential coefficients become infinite.

A function $f(x)$ which contains a radical may disappear in two different ways when a particular

*Williamson: Differential Calculus, p. 431.

value is attributed to the variable x; that is,

(1) because the quantity contained under the radical vanishes.

(2) because a factor with which the radical may be affected vanishes.

In the former case, the development according to Taylor's theorem can never agree with the function $f(x+h)$ for the particular value of x in question, for the reason already stated in the example.

In the latter case this is not the true, because the factor with which the radical is affected, and which becomes zero in the function, may cease to affect the radical in the differential coefficients of higher orders. In fact the radical may not disappear at all, and the series may in consequence present the necessary number of values.

Example 3.*

Let $f(x) = (x-a)^m \sqrt{x-b}$

where m is a positive integer.

Here we have

$$f'(x) = m(x-a)^{m-1}\sqrt{x-b} + \frac{(x-a)^m}{2\sqrt{x-b}}.$$

$$f''(x) = m(m-1)(x-a)^{m-2}\sqrt{x-b} + \frac{m(x-a)^{m-1}}{\sqrt{x-b}} - \frac{(x-a)^m}{4(x-b)^{3/2}}$$

Each differentiation causes one of the factors $(x-a)^m$ to disappear in the first term. After m differentiations, these factors would entirely disappear, and consequently the supposition that $x=a$, in causing the first m derived functions to vanish, will leave the radical $\sqrt{x-b}$ to remain in all the others.

The following examples give illustrations of cases where Taylor's theorem does not hold because some

* Williamson: Differential calculus, p. 432.

of the terms of the development are infinite

Example 4.

$$f(x) = \log x$$
$$f'(x) = \frac{1}{x}$$
$$f''(x) = -\frac{1}{x^2}$$
$$f'''(x) = \frac{2!}{x^3}$$
$$f^{iv}(x) = -\frac{3!}{x^4}$$

$$\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4} + \cdots\cdots + \frac{(-1)^{n-1} h^n}{n x^n} + \cdots\cdots$$

$\log(0+h) =$ finite number.

$$\log 0 + \frac{h}{0} - \frac{h^2}{2 0^2} + \cdots\cdots + \frac{(-1)^{n-1} h^n}{n 0^n} + \cdots = \infty.$$

Therefore this expansion fails.

Example 5

$$f(x) = \cot x$$
$$f'(x) = -\operatorname{cosec}^2 x$$
$$f''(x) = 2 \operatorname{cosec}^2 x \cot x$$
$$f'''(x) = -4 \operatorname{cosec}^2 x \cot^2 x + 2 \operatorname{cosec}^4 x.$$

$$\cot(x+h) = \cot x - \operatorname{cosec}^2 x\, h - 2 \operatorname{cosec}^2 x \cot x \frac{h^2}{2!} + (-4 \operatorname{cosec}^2 x \cot^2 x + 2 \operatorname{cosec}^4 x) \frac{h^3}{3!} + \cdots\cdots$$

$\cot(0+h) =$ finite number

When $x=0$, the above expansion becomes equal to infinity. Therefore the expansion does not hold at this point.

Example 6.

$f(x) = \operatorname{cosec} x$

$f'(x) = -\operatorname{cosec} x \cot x$

$f''(x) = \operatorname{cosec} x \cot^2 x + \operatorname{cosec}^3 x$

$f'''(x) = -\operatorname{cosec} x \cot^3 x - 2\operatorname{cosec}^3 x \cot x - 3\operatorname{cosec}^3 x \cot x$

$$\operatorname{cosec}(x+h) = \operatorname{cosec} x - \operatorname{cosec} x \cot x\, h + (\operatorname{cosec}^2 x \cot^2 x + \operatorname{cosec}^2 x)\frac{h^2}{2!} + (-2\operatorname{cosec}^2 x \cot^3 x - 2\operatorname{cosec}^4 x \cot x - 3\operatorname{cosec}^2 x \cot x)\frac{h^3}{3!} + \cdots$$

$\operatorname{cosec}(0+h) =$ finite number

When $x=0$, the expansion becomes infinite. Therefore this expansion fails.

Example 7.

$f(x) = a^{1/x}$

$f'(x) = -\log a \cdot a^{1/x} \frac{1}{x^2}$

$f''(x) = (\log a)^2 a^{1/x} \frac{1}{x^4} + \log a \cdot a^{1/x} \frac{2}{x^3}$

$$f'''(x) = -(\log a)^3 a^{\frac{1}{x}} \frac{1}{x^6} - (\log a)^2 a^{\frac{1}{x}} \frac{4}{x^5} - (\log a)^2 a^{\frac{1}{x}} \frac{2}{x^5} - \log a \cdot a^{\frac{1}{x}} \frac{3!}{x^4}$$

$$a^{\frac{1}{x+h}} = a^{\frac{1}{x}} + (-\log a \cdot a^{\frac{1}{x}} \frac{1}{x^2}) h + \left[(\log a)^2 \cdot a^{\frac{1}{x}} \frac{1}{x^4} + \log a \cdot a^{\frac{1}{x}} \frac{2}{x^3}\right] \frac{h^2}{2!} + \left[(-\log a)^3 \cdot a^{\frac{1}{x}} \frac{1}{x^6} - (\log a)^2 \cdot a^{\frac{1}{x}} \frac{6}{x^5} - \log a \cdot a^{\frac{1}{x}} \frac{3!}{x^4}\right] \frac{h^3}{3!} + \cdots$$

$a^{\frac{1}{0+h}}$ = a finite number

Since some of the terms of the expansion become infinite when $x = 0$, then Taylor's series fails in this case.

Example 7.

$$f(x) = \text{vers}^{-1} x$$

$$f'(x) = \frac{1}{\sqrt{2x - x^2}}$$

$$f''(x) = -\frac{1-x}{(2x-x^2)^{\frac{3}{2}}}$$

$$\therefore \text{vers}^{-1}(x+h) = \text{vers}^{-1} x + \frac{1}{\sqrt{2x-x^2}} h - \frac{1-x}{(2x-x^2)^{\frac{3}{2}}} \frac{h^2}{2!} + \cdots$$

$\text{vers}^{-1}(0+h)$ = finite number.

Since some of the terms of the above expansion become infinite when $x = 0$, then Taylor's theorem will not hold in this case.

Example 4.

$f(x) = \sin \frac{1}{x}$

$f'(x) = -\cos \frac{1}{x} \cdot \frac{1}{x^2}$

$f''(x) = -\sin \frac{1}{x} \cdot \frac{1}{x^4} + \cos \frac{1}{x} \cdot \frac{2}{x^3}$

$f'''(x) = \cos \frac{1}{x} \cdot \frac{1}{x^6} + \sin \frac{1}{x} \cdot \frac{4}{x^5} + \sin \frac{1}{x} \cdot \frac{3}{x^5}$
$- \cos \frac{1}{x} \cdot \frac{3!}{x^4}$

$\sin \frac{1}{x+h} = \sin \frac{1}{x} + \left(-\cos \frac{1}{x} \cdot \frac{1}{x^2}\right) h$
$+ \left(-\sin \frac{1}{x} \cdot \frac{1}{x^4} + \cos \frac{1}{x} \cdot \frac{2}{x^3}\right) \frac{h^2}{2!}$
$+ \left(\cos \frac{1}{x} \cdot \frac{1}{x^6} + \sin \frac{1}{x} \cdot \frac{4}{x^5} + \sin \frac{1}{x} \cdot \frac{3}{x^5}\right.$
$\left. - \cos \frac{1}{x} \cdot \frac{3!}{x^4}\right) \frac{h^3}{3!} + \cdots\cdots$

$\sin \frac{1}{0+h}$ = finite remainder.

Some of the terms of the above expansion become infinite when $x = 0$. Therefore in this case Taylor's theorem does not hold.

In the cases which we have considered, Taylor's expansion did not fail because some or all of the derivatives of the function became infinite. There are however, functions which have derivatives of every finite

consider the case where a function of a real variable x may be defined by means of analytical expressions, capable of being differentiated an indefinite number of times, which in the neighborhood of a point x_0 will be represented by one analytical expression only on one side of this point, while on the other side the limit of the convergency of the series representing the desired function. We shall by way of illustration define a function, capable of being differentiated an unlimited number of times, which only on one side of a point x_0, say the zero point, vanishes identically, and on the other side is different from zero, without changing the continuity of the function and its derivations at the zero point.

* Pringsheim; Math. Annalen, Bd. 22, p. 57.

Consider the following function.

$$(1) \quad f(x) = e^{-\frac{1}{x^2}} = \sum_0^\infty \frac{(-1)^\nu}{\nu!} \frac{1}{1+a^\nu x} \qquad (a > 1)$$

at the right side of (1), the function is continuous, and by virtue of the following theorem may be differentiated term by term thus giving the right sided derivatives.

*If the series

$$S(x) = f_1(x) + \cdots + f_n(x) + \cdots$$

is convergent, and the series

$$\sigma(x) = f_1'(x) + \cdots + f_n'(x) + \cdots$$

is convergent in the interior of the interval E; and also if the functions $f_1'(x) \cdots f_n'(x) \cdots$ are continuous in the interior of the interval E', then $S(x)$ can have a derivative equal to $\sigma(x)$ in the interior of E.

$$f(0) = \sum_0^\infty \frac{(-1)^\nu}{\nu!} \frac{1}{1+a^\nu \cdot 0} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$$

*If $\xi(x) = e^x$, then $\xi'(x) = e^x$, $\xi''(x) = e^x$, ...

Jordan: Cours D'Analyse Vol. 1, p. 313

Then by Maclaurin's series

$$e^x = S(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\cdots$$

If we place $x = 1$, then we have

$$\frac{1}{e} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots\cdots$$

Then we have

(2) $\quad f(0) = S = 0 \quad f(0+\delta) = \frac{1}{e},$

which shows the continuity to the right of zero.

We have for the n^{th} derivative of $f(x)$

$$f^n(x) = (-1)^n n! \sum_0^\infty \frac{(-1)^\nu}{\nu!} \frac{a^{n\nu}}{(1+a^\nu x)^{n-1}}$$

Let us denote the n^{th} derivative to the right of zero by $f_+^n(0)$, and then we have

$$f_+^n(0) = (-1)^n n! \sum_0^\infty \frac{(-1)^\nu}{\nu!} a^{n\nu}$$

We will then have according to (2)

(3) $\quad f_+^n(0) = f^n(+0) = (-1)^n n! \left(\frac{1}{e}\right)^{a^n}$

However, $f(x)$ will not represent

through the constantly convergent Maclaurin's series, the function

$$(4) \qquad \Psi(x) = \sum_{0}^{\infty \nu} (-1)^{\nu} \left(\frac{1}{\varepsilon}\right)^{a^{\nu}} \cdot x^{\nu}$$

As a consequence of this, the expression

$$(5) \qquad \phi(x) = \Psi(x) - f(x)$$

together with its right-sided derivatives vanishes for $x = 0$; also the expression (5), together with its right-sided derivatives, is finite and continuous in the neighborhood to the right of the zero point without vanishing identically. Let us take a function $F(x)$, which for $x \leqq 0$, has the value zero, and for $x \geqq 0$ has the value $\phi(x)$, and let this function $F(x)$ together with its derivatives be continuous. We shall now try to represent the function $F(x)$ for the interval $(-c, +c)$ by an analytical expression, say by the

Fourier's series. $F(-\epsilon) = 0$ and $F(+\epsilon) = \phi(\epsilon)$; that is, it has a finite value different from zero, and so in the limit the function $F(x)$ does not converge uniformly, and can not be differentiated term by term. Therefore $F(x)$ can in no case serve as an analytical expression of the derivatives of $F(x)$.

In order to overcome this difficulty, we will change the function $F(x)$ in the following manner. If we place

$$\phi(1-x) = \psi(1-x) - f(1-x), \tag{6}$$

then this expression behaves exactly the same on the left side of $x = 1$, as $\phi(x)$ did on the right of zero, that is

$$\phi(1-0) = 0 \quad , \quad \phi^n(1-0) = 0 \tag{7}$$

If we define the function $F(x)$ in the following manner:

$$\begin{cases} F(x) = 0 & \text{for } -1 \leqq x \leqq 0 \\ F(x) = \phi(x) \cdot \phi(1-x) & \text{for } 0 \leqq x \leqq +1, \end{cases} \tag{8}$$

then $F(x)$ so defined for the interval $-1 \leqq x \leqq +1$, will not only be continuous with all its derivatives, but will also possess the property that

$$F(-1+0) = F(1-0) = 0$$
$$F^n(-1+0) = F^n(1-0) = 0.$$

Since $F(x)$ and $F^n(x)$ have the same values for every subinterval, then they can be developed in a uniformly convergent Fourier's series. Moreover $F(x)$ can then be represented by a Fourier's series, capable of being differentiated term by term an unlimited number of times. Then we have

$$(9) \quad F(a) = \tfrac{1}{2} a_0 + \sum_1^\infty (a_\nu \cos \nu\pi x + b_\nu \sin \nu\pi x)$$

where

$$(10) \quad \left.\begin{matrix} a_\nu \\ b_\nu \end{matrix}\right\} = \int_0^1 \phi(x) \cdot \phi(1-x) \cdot \left\{\begin{matrix} \cos \\ \sin \end{matrix}\right\} \nu\pi x \cdot dx.$$

Now if $P(x)$ denotes a converging power series for the interval about zero, then the analytical expression

(11) $$F_1(x) = F(x) + P(x),$$

is finite and continuous in the interval about zero. It possesses derivatives, but represents the power series $P(x)$ only to the left of zero.

If we denote $P(x)$ by a converging power series, which like $F(x)$ has the periodicity 2, that is $P(x) = \cos \pi x$, then for every finite x, the continuous expression

(12) $$F_1(x) = F(x) + \cos \pi x$$

with all of its derivatives equal $\cos \pi x$, only in all the intervals, which are bounded on the left by an odd number, and on the right by an even number. Therefore, we see that two continuous analytical functions with all their derivatives need not be in any way identical, even if they are the same for an infinite number of intervals.

§ 2. Necessary and sufficient condition for the existence of Taylor's Theorem:- We are now going to consider the necessary and sufficient condition for the existence of Taylor's theorem, but in order to prove this theorem we will need to use two lemma which we will now prove.

Lemma 1. *Necessary condition for the existence of Taylor's theorem. Let $\sum_0^\infty c_\nu h^\nu$ denote a convergent series for all positive values of $h < R$. Having given a, an arbitrary value, let

(1) $\sum_0^\infty c_\nu (x-a)^\nu = f(x)$ for $0 \leqq x-a < R$.

Then $f(x)$ is a single valued, finite, and definite function within the interval $a \leqq x < a+R$. At $x = a$, this function has a right sided derivative, and for the interval $a < x < a+R$, it has

*Pringsheim: Math. Annalen, Vol. 44 p. 62.

a completely defined, finite derivative of every finite order, which for every finite integral μ (including $\mu = 0$) satisfies the condition

(2) $\tilde{n} \doteq \infty \; F_{\pm\mu, n}(h, k) = \tilde{n} \doteq \infty \; \frac{1}{(n \pm \mu)!} f^{n}(a+h) \cdot K^{n \pm \mu} = 0$

This holds for all values (h, k), which obey the law

$$0 \leqq h \leqq h + k \leqq r \quad \text{where } r < R.$$

Proof: Since (1) is a convergent power series, it can converge to but one finite value. Therefore the $f(x)$ is single valued and finite in the interval $a \leqq x < a + R$. * By term by term differentiation of (1), we have

(3) $$\sum_{\mu}^{\infty \nu} \nu(\nu-1) \cdots\cdots (\nu - \mu + 1) c_{\nu} (x-a)^{\nu - \mu} = f^{\mu}(x).$$

The summation in (3) equals $f^{\mu}(x)$, because since it is a power series, it is uniformly convergent, and each term is continuous.

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* This follows by virtue of the theorem on p. 87 of this thesis.

According to (3), it is seen that the derivatives of $f(x)$ exist on the interval

$$0 \leqq x - a < R.$$

It follows from (1) and (3) that when $x = a$, then

$$c_0 = f(a), \quad c_p = \tfrac{1}{p!} f^p(a),$$

and that when $x - a = h$, the equations (1) and (3) may be written in the following form

$$\left.\begin{aligned} (4)\quad & f(a+h) = \sum_0^\infty{}^{\nu} \frac{1}{\nu!} f^\nu(a) \cdot h^\nu \\ (5)\quad & f^p(a+h) = \sum_p^\infty{}^{\nu} \frac{1}{(\nu-p)!} f^\nu(a) h^{\nu-p} \end{aligned}\right\} \text{for } 0 \leqq h < R$$

Let us take arbitrarily the function

$$(6)\qquad \Phi(x) = \sum_0^\infty{}^{\nu} |c_\nu| (x-a)^\nu,$$

so that for the interval $a \leqq x < a+R$, $\Phi(x)$ is a completely defined function of a similar character as $f(x)$, and for $x - a = h$, we have

$$(7)\quad \Phi(a+h) = \sum_0^\infty{}^{\nu} |c_\nu| h^\nu = \sum_0^\infty{}^{\nu} \frac{1}{\nu!} \Phi^\nu(a) h^\nu$$

$$(8)\quad \Phi^p(a+h) = \sum_p^\infty{}^{\nu} \frac{\nu!}{(\nu-p)!} |c_\nu| h^{\nu-p} = \sum_0^\infty{}^{\nu} \frac{1}{(\nu-p)!} \Phi^\nu(a) h^{\nu-p}.$$

since $\phi(a+h)$ and $\phi'(a+h)$ are uniformly convergent power series, they must be continuous in h for the interval $0 \leqq h < R$, and that for every p however large:

(9) $\quad |f^p(a+h)| \leqq \phi^p(a+h).$

$|f^p(a+h)|$ is less than $\phi^p(a+h)$ when the c_ν in $f(x)$ contains negative terms, and is equal to $\phi^p(a+h)$ when c_ν in $f(x)$ contains all positive terms.

It follows that for the proof of (2), it is sufficient to show that the analogous relation exist for $\phi^n(a+h)$.

We have for $0 \leqq h \leqq h+k < R$

$$\phi(a+h+k) = \sum_0^\infty{}^\nu \frac{1}{\nu!} \phi^\nu(a) \cdot (h+k)^\nu$$

or, if we arrange this equation according to powers of k, we have by referring to (7)

(10) $\quad \phi(a+h+k) = \sum_0^\infty{}^\nu \frac{1}{\nu!} \phi^\nu(a+h) \cdot k^\nu$

Of this series, it is seen that according to the assumption of $r < R$,

it converges uniformly for all values of (h, k) which satisfy the condition that $0 \leqq h \leqq h+k \leqq r$. We may rearrange (9) in the following manner.

$$\phi(a+h+k) = \sum_{\nu}^{n-1} \frac{1}{\nu!} \phi^{\nu}(a+h).k^{\nu} + \sum_{n}^{\infty} \frac{1}{\nu!} \phi^{\nu}(a+h).k^{\nu}$$

$$= S_n(h,k) + T_n(h,k).$$

If we take h and k within the interval already indicated, and take δ, a positive value subject only to the condition that

$$h+k+2\delta < R;$$

moreover if we take ξ and η as two arbitrary numbers in the interval $(-1, +1)$, we have

$$\phi(a+h+k+(\xi+\eta)\delta) = \phi(a+(h+\xi\delta)+(k+\eta\delta)$$

$$= S_n(h+\xi\delta, k+\eta\delta) + T_n(h+\xi\delta, k+\eta\delta).$$

Transposing, we have

$$T_n(h+\xi\delta, k+\eta\delta) = \phi(a+h+k+(\xi+\eta)\delta) - S_n(h+\xi\delta, k+\eta\delta).$$

By adding a certain value to the right hand side of the above equation, and subtracting the same value from it does not change the value of the equation. Hence we have

$$T_n(h+\xi\delta, k+\eta\delta) = \phi(a+h+k+(\xi+\eta)\delta) - \phi(a+h+k) - S_n(h+\xi\delta, k+\eta\delta) + S_n(h,k) + T_n(h,k).$$

Then we certainly have

$$(11)\quad T_n(h+\xi\delta, k+\eta\delta) \leqq |\phi(a+h+k+(\xi+\eta)\delta) - \phi(a+h+k)| + |S_n(h+\xi\delta, k+\eta\delta) - S_n(h,k)| + T_n(h,k).$$

If we suppose ϵ to be an arbitrarily small positive value, we can so take n that for an arbitrarily chosen but definite pair of values (h, k)

$$T_n(h,k) < \epsilon/3$$

Since $\phi(a+h+k)$ and $S_n(h,k)$ are uniformly convergent power series, then $\phi(a+h+k)$ is a continuous

function of $(h+k)$, and $S_n(h,k)$ is a continuous function of h and k. Then a definite value δ may be so taken that

$$|\phi(a+h+k+(\xi+\eta)\delta) - \phi(a+h+k)| < \epsilon/3$$

$$|S_n(h+\xi\delta, k+\eta\delta) - S_n(h,k)| < \epsilon/3.$$

By adding these three inequalities, we have

$$T_n(h+\xi\delta, k+\eta\delta) < \epsilon,$$

and since the series with which we are concerned contains only positive terms, we have

(12) $\quad T_\nu(h+\xi\delta, k+\eta\delta) < \epsilon \quad (\text{for } \nu \geq n).$

That is series (10), considered as a function of h and k, converges uniformly for a known interval about every single point (h,k) which belongs in the interval $0 \leq h \leq h+k \leq r$, where $r < R$.

* If a given series converges uniformly

* Weierstrass: Abhandlungen aus der Functionslehre § 71, 72.

in an interval at every point which lies within the interval, or on the boundary of this interval, then it converges uniformly in the entire interval. We shall show that the series (10) converges uniformly for every interval, so that when r is taken as small as we please and always less than R, n is so fixed that

$$T_\nu(h,k) < \epsilon, \tag{13}$$

for $\nu \geq n$, and for all positive (h,k) which satisfies the condition $h+k \leq r$. This analogy holds for every series of the following form

$$\sum_0^\infty \frac{1}{(\nu+\mu)!}\, \phi^{\nu}(a+h).k^{\nu+\mu},$$

and also for every series which results from (10) through μ differentiations with respect to k, namely;

$$\sum_0^\infty \frac{1}{(\nu-\mu)!}\, \phi^{\nu}(a+h).k^{\nu-\mu}.$$

since these series contain only positive terms, we have

$$\frac{1}{(\nu \pm p)!}\, \phi^{\nu}(a+h).\, k^{\nu \pm p} < \epsilon,$$

for $\nu \geqq n$ and $0 \leqq h \leqq h+k \leqq r$

since by (9)

$$|f^{p}(a+h)| \leqq \phi^{p}(a+h), \text{ then}$$

$$\frac{1}{(n \pm p)!}\, f^{n}(a+h).\, k^{\nu \pm p} < \epsilon,$$

which is the same as saying

$$\lfloor_{n=\infty} \frac{1}{(n \pm p)!}\, f^{n}(a+h).\, k^{n \pm p} = 0.$$

Lemma 2.* If $f^{n}(x)$ is defined for every positive integer n, and for $a \leqq x < a+R$, and is a function which for a definite, positive or negative integral p satisfies the condition.

$$\lfloor_{n=\infty} F_{p,n}(h,k) = \lfloor_{n=\infty} \frac{1}{(n+p)!}\, f^{n}(a+h).\, k^{n+p} = 0$$

and this for all (h,k) which obey the law; $0 \leqq h \leqq h+k < r$, where $r < R$,

* Pringsheim: Math. Annalen, Vol. 44, p. 67.

then the same holds for every integral p.

Proof: We have

$$F_{p+1,\nu}(h,k) = \frac{1}{(\nu+p+1)!} f^{\nu}(a+h) k^{\nu+p+1} = \frac{k}{\nu+p+1} F_{p,\nu}(h,k),$$

and for every $k \leqq r$

$$|F_{p+1,\nu}(h,k)| < |F_{p,\nu}(h,k)|,$$

where $\nu+p-1 > r$. Since $\overset{\frown}{n} \doteq \infty\, F_{p,n}(h,k)$ vanishes in the given interval, and since

$$|F_{p+1,\nu}(h,k)| < |F_{p,\nu}(h,k)|,$$

then $\overset{\frown}{n} \doteq \infty\, F_{p+1,\nu}(h,k)$ and in general for $q > 0$, $\overset{\frown}{n} \doteq \infty\, F_{p+q,\nu}(h,k)$ also vanishes.

In order to prove that $\overset{\frown}{n} \doteq \infty\, F_{p-q,\nu}(h,k)$ vanishes, we have

$$F_{p-1,\nu}(h,k) = \frac{1}{(\nu+p-1)!} f^{\nu}(a+h) \cdot k^{\nu+p-1}$$

$$= \frac{1}{(\nu+p)!} f^{\nu}(a+h) \cdot (k+\delta)^{\nu+p} \cdot \frac{k^{\nu+p-1}}{(k+\delta)^{\nu+p}} \cdot (\nu+p)$$

$$= F_{p,\nu}(h, k+\delta) \frac{k^{\nu+p-1}}{(k+\delta)^{\nu+p-1}} \cdot \frac{\nu+p}{k+\delta}$$

If we take $r < R$, then we may so choose a positive number δ, that

$r+\delta < R$. We have for every positive value of $k \leqq r$

$$(1)\quad \left| F_{p-1,\nu}(h,k) \right| < F_{p,\nu}(h,k+\delta)\left(\frac{r}{r+\delta}\right)^{\nu+p-1}\cdot\frac{\nu+p}{\delta}$$

If we take a positive value n so that $\nu \geqq n$, we have

$$(2)\quad \left| F_{p,\nu}(h,k+\delta) \right| < \epsilon \quad , \text{ for } 0 \leqq h \leqq h+k \leqq r.$$

Hence

$$\sum_{\nu=\infty}^{\nu} \left(\frac{r}{r+\delta}\right)^{\nu+p-1}\frac{\nu+p}{\delta}$$

$$= \sum_{\nu=\infty}^{\nu} \frac{\nu}{\delta}\left(\frac{r}{r+\delta}\right)^{\nu+p-1} + \sum_{\nu=\infty}^{\nu} \frac{p}{\delta}\left(\frac{r}{r+\delta}\right)^{\nu+p-1}$$

$$\sum_{\nu=\infty}^{\nu} \frac{p}{\delta}\left(\frac{r}{r+\delta}\right)^{\nu+p-1} = 0$$

$$\sum_{\nu=\infty}^{\nu} \frac{\nu}{\delta}\left(\frac{r}{r+\delta}\right)^{\nu+p-1} = \frac{1}{\delta}\left(\frac{r}{r+\delta}\right)^{p-1}\sum_{\nu=\infty}^{\nu}\left(\frac{r}{r+\delta}\right)^{\nu}$$

$$= \frac{1}{\delta}\left(\frac{r}{r+\delta}\right)^{p-1}\sum_{\nu=\infty}^{\nu} \nu\left(\frac{1}{1+\frac{\delta}{r}}\right)^{\nu}$$

Let $\left(1+\frac{\delta}{r}\right) = a$.

Then we have

$$\frac{1}{\delta}\left(\frac{1}{a}\right)^{p-1}\sum_{\nu=\infty}^{\nu}\frac{\nu}{a^{\nu}} = \frac{1}{\delta}\left(\frac{1}{a}\right)^{p-1}\sum_{\nu=\infty}^{\nu}\frac{1}{\frac{a^{\nu}}{\nu}} =$$

$$\frac{1}{\delta}\left(\frac{1}{a}\right)^{p-1}\frac{1}{\sum_{\nu=\infty}^{\nu}\frac{a^{\nu}}{\nu}} = \frac{1}{\delta}\left(\frac{1}{a}\right)^{p-1}\frac{1}{\infty} = \frac{1}{\delta}\left(\frac{1}{a}\right)^{p-1}0 = 0.$$

$$\therefore\quad (3)\quad \left(\frac{r}{r+\delta}\right)^{\nu+p-1} \leqq 1.$$

Therefore, according to equations (1), (2), and (3)

$$|F_{p-1,\nu}(h,k)| < \epsilon$$

It follows that in general for any q we have

$$\lim_{n=\infty} F_{p-q,n}(h,k) = 0.$$

The Necessary and sufficient Condition for the Existence of Taylor's Theorem.* The necessary and sufficient conditions that in the interval $0 \leqq h < R$, the single valued function $f(x+h)$ may be representable by Taylor's theorem

$$f(a+h) = \sum_0^\infty \frac{1}{\nu!} f^\nu(a) h^\nu$$

are (1) that $f(x)$ for the interval $a \leqq x < a+R$ shall be finite, and for $x=a$ possess a right-sided derivative, and for $a < x < a+R$ possess fully defined derivatives of every finite order.

(2) That for every positive or negative integral value p (including $p=0$), we have

$$\lim_{n=\infty} F_{p,n}(h,k) = \lim_{n=\infty} \frac{1}{(n+p)!} f^n(a+h) \cdot k^n = 0$$

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* Pringsheim: Math. Annalen, Vol. 44, p. 68.

Moreover for all values (h, k) which obey the law, that we have

$$0 \leqq h \leqq h+k \leqq r \text{ , where } r < R$$

Proof: Since the sum of the series $\sum \frac{1}{v!} f^{v}(a) h^{v}$ for $0 \leqq h < R$ is definite and single valued, and since it is a power series, then the series converges in the interval $0 \leqq h < R$, and possesses a finite value. The proof that these are necessary conditions is the same as in Lemma 1.

In order to prove that these are sufficient conditions, we see that to Lemma 2 if $\sum_{n=0}^{n=\infty} F_{p,n}(h, k) = 0$ for any particular value of p, that it is zero for every other p. We shall now show that $\sum_{n=0}^{n=\infty} F_{p,n}(h, k)$ vanishes when $p = 0$ and when $p = -1$, from which we may conclude that it vanishes for every p.

Let us consider the following cases:

(A) when $p = 0$

(B) when $p = -1$

(A) If we take a positive value r as small as we please but always less than R, then there will always an infinitely large number of positive integral numbers m, so that

$$(1) \qquad r + \frac{R}{m} < R$$

When $p = 0$, we have

$$(2) \quad \sum_{n=0}^{n=\infty} F_{0,n}(h, k) = \sum_{n=0}^{n=\infty} \frac{1}{n!} f^n(a+h) \cdot k^n = 0$$

This holds for all values of (h, k) where

$$0 \leqq h \leqq h + k \leqq r + \frac{R}{m}$$

Then we have

$$(3) \quad \sum_{n=0}^{n=\infty} \frac{1}{n!} f^n(a) \left(r + \frac{R}{m}\right)^n = 0$$

$$(4) \quad \sum_{n=0}^{n=\infty} \frac{1}{n!} f^n(a+h) \left(\frac{R}{m}\right)^n = 0$$

} for $0 \leqq h < r$

Using* Lagrange's remainder we may write

$$(5) \quad f(a+h) = \sum_{0}^{n-1} \frac{1}{v!} f^v(a)\, h^v + \frac{1}{n!} f^n(a + \theta h) \cdot h^n$$

for $0 < \theta < 1$,

and when $m > 2$, it follows from (4) that

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* Page 10 of this thesis

(6) $\lim_{n=\infty} \frac{1}{n!} f^n(a+\theta\frac{R}{m})(\frac{R}{m})^n = 0$, for $0<\theta<1$

From (6) we see that the limit of the remainder after n terms is zero, and hence according to (5) we conclude that

(7) $f(a+h) = \sum_0^\infty \frac{1}{\nu!} f^\nu(a)\, h^\nu = S(a,h)$,

for $0 \leqq h \leqq \frac{R}{m}$ and in general for $0 \leqq h \leqq \frac{r}{m}$. When $h = \frac{r}{m}$ lies within the interval for which (7) holds, and in the interval of convergence of $S(a,h)$, then not only

(8) $f(a+\frac{r}{m}) = \sum_0^\infty \frac{1}{\nu!} f^\nu(a)(\frac{r}{m})^\nu$,

but also by term by term differentiation, we obtain

(9) $f^\lambda(a+\frac{r}{m}) = \sum_\lambda^\infty \frac{1}{(\nu+\lambda)!} f^\nu(a)\left(\frac{r}{m}\right)^{\nu-\lambda}$

In order to extend the interval for which the equation (7) holds, we have, going on from the point $(a+\frac{r}{m})$ according to the analogy of (5)

(10) $f(a+\frac{r}{m}+k) = \sum_0^{n-1} \frac{1}{\nu!} f^\nu(a+\frac{r}{m})\cdot k^\nu$

$+ \frac{1}{n!} f^n(a+\frac{r}{m}+\theta k)\cdot k^n$

According to (4), we know that the remainder vanishes for $k \leq \frac{R}{m}$, when $m > 3$. It follows that

(11) $$f(a + \frac{r}{m} + k) = \sum_0^\infty \frac{1}{\nu!} f^\nu(a + \frac{r}{m}) \cdot k^\nu$$
$$= S(a + \frac{r}{m}, k),$$

for $0 < k < \frac{R}{m}$

$$S(a, \frac{r}{m} + k) = \sum_0^\infty \frac{1}{\nu!} f^\nu(a) (\frac{r}{m} + k)^\nu =$$
$$f(a) + f'(a)(\frac{r}{m} + k) + \frac{1}{2!} f''(a)(\frac{r}{m} + k)^2$$
$$+ \frac{1}{3!} f'''(a)(\frac{r}{m} + k)^3 + \frac{1}{4!} f^{IV}(a)(\frac{r}{m} + k)^4 + \cdots$$
$$= f(a) + f'(a)(\frac{r}{m}) + f'(a)\, k + \frac{1}{2!} f''(a)(\frac{r}{m})^2$$
$$+ \frac{1}{2!} f''(a)(2\frac{r}{m} \cdot k) + \frac{1}{2!} f''(a)(k)^2 + \cdots\cdots$$

As a result of condition (3) where $k \leq \frac{R}{m}$, this absolutely convergent series can be arranged according to powers of k. This gives

$$S(a, \frac{r}{m} + k) = \sum_0^\infty \lambda \left\{ \sum_0^\infty \frac{1}{(\nu - \lambda)!} f^\nu(a)(\frac{r}{m})^{\nu - \lambda} \right\} \frac{k^\lambda}{\lambda}$$
$$= \sum_0^\infty \frac{1}{\lambda!} f^\lambda(a + \frac{r}{m}) k^\lambda \quad \text{by (9)}$$
$$= S(a + \frac{r}{m}, k) \quad \text{for } 0 \leq k \leq \frac{R}{m}.$$

With the help of this, equation (11) now takes the following form:

(12) $$f(a + \frac{r}{m} + k) = S(a, \frac{r}{m} + k)$$

for $0 \leqq k \leqq \frac{R}{m}$

When we place $\frac{2r}{m} + k = h$, we have $f(a+h) = S(a, h)$, for $\frac{2r}{m} \leqq h \leqq \frac{2r}{m} + \frac{R}{m}$.

In an analogous manner $f(a + \frac{2r}{m} + k) = S(a + \frac{2r}{m}, k)$, for $0 \leqq k \leqq \frac{R}{m}$.

With the help of the equation

$$S(a, \tfrac{2r}{m} + k) = \sum_{0}^{\infty \nu} \frac{1}{\nu!} f^{\nu}(a) (\tfrac{2r}{m} + k)^{\nu} = S(a + \tfrac{2r}{m}, k),$$

we have

$$f(a + \tfrac{2r}{m} + k) = S(a, \tfrac{2r}{m} + k), \text{ for } 0 \leqq k \leqq \tfrac{R}{m}.$$

The equation (7) holds for the interval $0 \leqq h \leqq \frac{2r}{m} + \frac{R}{m}$, and in general for $0 \leqq h \leqq \frac{3r}{m}$. Through $(m-1)$ applications of this extension, we get the existence of (7) for $0 \leqq h \leqq r$, which we were to prove.

(B) When $p_1 = -1$

$$(13) \quad \sum_{n=0}^{\infty} F_{-1,n}(h, k) = \sum_{n=0}^{\infty} \frac{1}{(n-1)!} f^{n}(a+h) \cdot k^{n-1} = 0.$$

This holds for $0 \leqq h \leqq h + k \leqq r$,

where $r < R$.

If we write $\theta h = h$, and $(1-\theta) h = k$, then

$$(14) \quad \sum_{n=0}^{\infty} \frac{1}{(n-1)!} f^{n}(a + \theta h) \cdot (1-\theta)^{n-1} \cdot h^{n-1} = 0$$

for $0 \leqq \theta \leqq 1$ and $0 \leqq h \leqq r$.
With the help of Cauchy's remainder, for every $h \leqq r < R$, we have

$$(15)\quad f(a+h) = \sum_{0}^{n-1} \frac{1}{\nu!} f^{\nu}(a) h^{\nu} + \frac{1}{(n-1)!} f^{n}(a+\theta h)(1-\theta)^{n-1} h^{n-1}$$

Then according to (14)

$$(16)\quad f(a+h) = \sum_{0}^{\infty} \frac{1}{\nu!} f^{\nu}(a) \cdot h^{\nu}$$

for $0 \leqq h \leqq r < R$.

Since $n = \infty$ $F_{p,n}(h,k)$ vanishes for the particular values $p = 0$ and $p = -1$, it vanishes for every value of p. With this our demonstration is completed.

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